If $d=1$, i.e., $u: \Omega \subset \mathbb{R}^N \to \mathbb{R}$

$f: \mathbb{R}^N \to [-\infty, +\infty]$

then quasiconvexity at $\xi_0 \in \operatorname{int}(\operatorname{dom}_e f)$

$=$ convexity at ξ_0

In this scalar-valued case, swlsc of the functional is guaranteed without imposing growth conditions on f:
(nor convergence of gradients other than in $\mathcal{D}'$)

Thm

$\Omega \subset \mathbb{R}^N$ open, $|\Omega| < +\infty$, $1 \leq p \leq +\infty$

$f: \mathbb{R}^N \to (-\infty, +\infty]$ convex, lsc
bounded from below

Then if $u_n \to u$ in $L^1_{loc}(\Omega)$, $u_n, u \in V^{1,p}(\Omega; \mathbb{R})$

$\Downarrow$

$$\int_\Omega f(\nabla u)\, dx \leq \varliminf_{n \to +\infty} \int_\Omega f(\nabla u_n)\, dx$$

If f fails to be quasiconvex, then it may happen

$$\mathcal{F}(u) := \inf \left\{ \varliminf_{n \to +\infty} \int_\Omega f(\nabla u_n)\, dx : u_n \to u \right\} < \int_\Omega f(\nabla u)\, dx$$

Is it possible to obtain an integral representation

$$\mathcal{F}(u) = \int_\Omega \bar{f}(\nabla u)\, dx \, ?$$

If yes, then what is the relation between f and $\bar{f}$?

RELAXATION THEORY

$\Omega \subset \mathbb{R}^N$ open, $f: \mathbb{R}^{d \times N} \to [-\infty, +\infty]$ Borel measurable

$$I(u) := \int_\Omega f(\nabla u(x))\,dx, \qquad u \in W^{1,p}(\Omega; \mathbb{R}^d)$$

- $\mathcal{E} : W^{1,p}(\Omega; \mathbb{R}^d) \to [-\infty, +\infty]$ relaxed (or effective) energy ... is the greatest swlsc (w-$*$ $p = +\infty$) functional below I

- $\mathcal{H} : W^{1,p}(\Omega; \mathbb{R}^d) \to [-\infty, +\infty]$

$$u \mapsto \sup\{\Phi(u) : \Phi \leq I,\ \Phi \text{ sw } (w\text{-}*) \text{ lsc in } W^{1,p}\}$$

- $\mathcal{F} : W^{1,p}(\Omega; \mathbb{R}^d) \to [-\infty, +\infty]$

$$u \mapsto \inf_{\{u_n\}} \left\{ \underline{\lim}_{n \to +\infty} \int_\Omega f(\nabla u_n)\,dx : u_n \rightharpoonup u \text{ in } W^{1,p} \ (\overset{*}{\rightharpoonup}) \right\}$$

Rmk If $I(\cdot)$ is coercive, i.e.,

$$\boxed{\lim_{\|u\|_{W^{1,p}} \to +\infty} I(u) = +\infty}$$

and if $1 < p \leq +\infty$, then

$$\boxed{\mathcal{E} = \mathcal{H} = \mathcal{F}}$$

Recall $f : \mathbb{R}^N \to [-\infty, +\infty]$

conjugate (or polar) of f:

$$f^*(\xi) := \sup_{\eta \in \mathbb{R}^N} \{\xi \cdot \eta - f(\eta)\}$$

biconjugate (or bipolar) **of f**:

$$f^{**}(\xi) := (f^*)^*(\xi) = \sup_{\eta \in \mathbb{R}^N} \{\xi \cdot \eta - f^*(\eta)\}.$$

Rmk If $f : \mathbb{R}^N \to (-\infty, +\infty]$ is such that

$$f(\xi) \geq a + b \cdot \xi, \quad \text{some } a \in \mathbb{R},\ b \in \mathbb{R}^N$$

then

$$f^{**} = \operatorname{lsc}(Cf)$$

where, for $g : \mathbb{R}^N \to [-\infty, +\infty]$,

- $\operatorname{lsc} g$ ... lower semicontinuous envelope of g, i.e., the greatest lsc function below g
- $C g$ ... convex envelope of g, i.e., the greatest convex function below g

If $a + b \cdot \xi \leq f(\xi) < +\infty$ for all $\xi \in \mathbb{R}^N$ then

$$\boxed{f^{**} = \operatorname{lsc}(Cf) = C(\operatorname{lsc} f) = Cf}$$

let
- $C^1 f(v) := \inf \{ \theta f(v_1) + (1-\theta) f(v_2) : \theta \in [0,1], v_1, v_2 \in \mathbb{R}^N, v = \theta v_1 + (1-\theta) v_2 \}$
- $C^{n+1} f(v) := C^1(C^n f)(v), \quad n \in \mathbb{N}$

Lemma $Cf \leq C^{n+1} f \leq C^n f \leq f$ for all $n \in \mathbb{N}$

and

$$\boxed{\lim_{n \to +\infty} C^n f(v) = Cf(v)}, \quad \forall v \in \mathbb{R}^N.$$

Thm $\Omega \subset \mathbb{R}^N$ open, $|\Omega| < +\infty$, $1 \leq p < +\infty$

$f: \mathbb{R}^N \to \mathbb{R}$ Borel function.

Assume that there exist $C > 0$, $a \in \mathbb{R}$, $b \in \mathbb{R}^N$ s.t.

$$a + b \cdot \xi \leq f(\xi) \leq C(1 + |\xi|^p) \quad \forall \xi \in \mathbb{R}^N.$$

Then

$$\boxed{\mathcal{E}(u) = \int_\Omega f^{**}(\nabla u)\, dx}$$

and if $1 < p < +\infty$ then

$$\boxed{\int_\Omega f^{**}(\nabla u)\,dx = \inf_{\{u_n\}} \left\{ \varliminf_{n \to +\infty} \int_\Omega f(\nabla u_n)\,dx : u_n \rightharpoonup u \text{ in } W^{1,p} \right\}.}$$