Thm

$\Omega \subset \mathbb{R}^N$ open, $|\Omega| < +\infty$, $1 \leq p \leq +\infty$

$f : \mathbb{R}^{d \times N} \to \mathbb{R}$ quasiconvex

$$\boxed{(*)\ \bullet \text{ if } 1 < p < +\infty \text{ then } \lim_{|\xi| \to +\infty} \frac{f^-(\xi)}{|\xi|^p} = 0}$$

and

$$f(\xi) \leq C(1 + |\xi|^p), \quad \text{some } C > 0$$

- if $p = 1$ then $|f(\xi)| \leq C(1 + |\xi|)$, some $C > 0$.

Then $u \in V^{1,p}(\Omega; \mathbb{R}^d) \mapsto \int_\Omega f(\nabla u)\,dx$ is swlsc in $V^{1,p}$ (w-$*$ if $p = +\infty$).

Rmk $(*)$ ensures that if $u_n \rightharpoonup u$ $V^{1,p}$ then $\{f^-(\nabla u_n)\}$ is equi-integrable (no concentrations)

A (possibly) more general condition than $(*)$ was derived by S. Krömer:

$\Omega \subset \mathbb{R}^N$ open bounded, $1 < p < +\infty$

$f : \mathbb{R}^{d \times N} \to \mathbb{R}$ continuous, $|f(\xi)| \leq C(1 + |\xi|^p)$

Then $u \in V^{1,p}(\Omega; \mathbb{R}^d) \mapsto \int_\Omega f(\nabla u)\,dx$ is swlsc in $V^{1,p}$

iff

- f is quasiconvex
- $$\boxed{\forall\, \nu \in S^{N-1} \ \forall\, \varepsilon > 0 \ \exists\, C_\varepsilon > 0 \text{ s.t. } \int_{B_\nu^+(0,1)} f(\nabla \varphi)\,dx \geq -\varepsilon \int_{B_\nu^+(0,1)} |\nabla \varphi|^p\,dx - C_\varepsilon}$$

for all $\varphi \in C_c^\infty(B(0,1); \mathbb{R}^d)$, with

$$B_\nu^+(0,1) := \{x \in B(0,1) : x \cdot \nu > 0\}.$$

Thm

$\Omega \subset \mathbb{R}^N$ open, $|\Omega| = +\infty$, $1 \leq p \leq +\infty$

$f : \mathbb{R}^{d \times N} \to \mathbb{R}$ quasiconvex

- $1 < p < +\infty$ then $0 \leq f(\xi) \leq C(1 + |\xi|^p)$ some $C > 0$
- $p = 1$ then $-c|\xi| \leq f(\xi) \leq C(1 + |\xi|)$ some $C > 0$
- $p = +\infty$ then $f \geq 0$

Then $u \in V^{1,p}(\Omega; \mathbb{R}^d) \mapsto \int_\Omega f(\nabla u)\,dx$ is sulsc in $V^{1,p}$
(w-* if $p = +\infty$)