Last time: $\Omega \subset \mathbb{R}^N$ open, $|\Omega| < +\infty$, $1 < p < +\infty$

$0 \leq f(\xi) \leq C(1 + |\xi|^p)$

$u_n \rightharpoonup u$ in $V^{1,p}(\Omega; \mathbb{R}^d)$

$\lim_{n \to +\infty} \int_\Omega f(\nabla u_n)\, dx$ exists and is finite.

Want to find $\{w_n\}$ s.t. :

- $w_n \rightharpoonup u$ in $V^{1,p}(\Omega; \mathbb{R}^d)$
- $w_n = u$ near $\partial\Omega$

  $w_n = u_n$ away from $\partial\Omega$
- $\overline{\lim_{n \to +\infty}} \int_\Omega f(\nabla w_n)\, dx \leq \lim_{n \to +\infty} \int_\Omega f(\nabla u_n)\, dx$

It is tempting to try the following:

$$\begin{cases} \varphi_k \in C_c^\infty(\Omega_{1/k}; [0,1]) \\ \varphi_k = 1 \quad \text{in } \Omega_{2/k} \\ \|\nabla \varphi_k\|_\infty \leq Ck \end{cases}$$

$$\Omega_{1/k} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/k\} \cap B(0,k)$$

Set $\boxed{w_{k,n} := \varphi_k u_n + (1 - \varphi_k) u.}$

Then $\begin{cases} w_{k,n} = u & \text{if } x \notin \Omega_{1/k} \\ w_{k,n} = u_n & \text{if } x \in \Omega_{2/k} \end{cases}$

- $\overline{\lim_k}\, \overline{\lim_n} \|w_{k,n} - u\|_{L^p(\Omega; \mathbb{R}^d)} = 0$
- $\overline{\lim_k}\, \overline{\lim_n} \|\nabla w_{k,n}\|_{L^p(\Omega; \mathbb{R}^{d \times N})} \leq \overline{\lim_n} \|\nabla u_n\|_{L^p} + \|\nabla u\|_{L^p} < +\infty$

and

$$\overline{\lim_{k}}\ \overline{\lim_{n}} \int_{\Omega} f(\nabla w_{k,n}) \leq \lim_{n} \int_{\Omega} f(\nabla u_n)\, dx + \overline{\lim_{k}}\ \overline{\lim_{n}} \int_{\Omega_{1/k} \setminus \Omega_{2/k}} |\nabla u_n|^p \, dx$$

$\{|\nabla u_n|^p\}$ is not necessarily equi-integrable, so it may be

$$\overline{\lim_{k}}\ \overline{\lim_{n}} \int_{\Omega_{1/k} \setminus \Omega_{2/k}} |\nabla u_n|^p \, dx > 0$$

and thus we cannot use a diagonalization argument to construct the desired $\{w_n\}$ as

$$w_n := w_{k(n),n} \quad \text{or} \quad w_k := w_{k,n(k)}.$$

We had two different ways to tackle this problem:

1) Use De Giorgi's Slicing Method to determine "explicitly" a layer "$\Omega_{1/k} \setminus \Omega_{2/k}$" on which the transition of the cut-off function φ between 1 and 0 could be made with least energy.
No need to extract a subsequence!

2) Use a measure theoretic approach: up to a subsequence,
$$|\nabla u_n|^p \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \mu$$
μ nonnegative, finite Radon measure
choose $t_k \searrow 0^+$ s.t. $\mu(\partial \Omega_{t_k}) = 0$
perform the transition of φ from 1 to 0 nearby $\partial\Omega_{t_k}$.

And now:

3) If the problem is possible lack of equi-integrability of $\{|\nabla u_n|^p\}$, then get rid of it from the start!

will need to extract a subsequence!

Use the Decomposition Lemma:

Ω open, $1 < p < +\infty$

$\exists \{u_{n_i}\} \subset \{u_n\}$ s.t.

$$\begin{cases} u_{n_i} = (u + \bar{w}_i) + v_i \\ \bar{w}_i \in W^{1,p}(\Omega) \cap W^{1,\infty}, \quad \bar{w}_i \rightharpoonup 0 \text{ in } W^{1,p} \\ w_i := u + \bar{w}_i, \quad w_i \rightharpoonup u \text{ in } W^{1,p} \\ \qquad \{|\nabla w_i|^p\} \text{ equi-integrable} \\ |\{x \in \Omega \cap B(0,R) : v_i(x) \neq 0\}| \to 0 \text{ for all } R > 0 \end{cases}$$

If Ω is bounded then can impose $w_i = u$ nearly $\partial\Omega$, and

$$\overline{\lim_i} \int_\Omega f(\nabla w_i)\,dx \leq \lim_n \int_\Omega f(\nabla u_n)\,dx$$

$$+ \overline{\lim_i} \int_{\Omega \cap \{v_i \neq 0\}} C(1 + |\nabla w_i|^p) = 0$$

by equi-integrability

What if $p=+\infty$ and $|\Omega|=+\infty$?

Here f is locally bounded and $\int_\Omega f(\nabla u)\,dx \in \mathbb{R}$.

Start as before:
$$
\begin{cases}
\varphi_k \in C^\infty_c(\Omega_{1/k};[0,1]) \\
\varphi_k \equiv 1 \quad \text{in } \Omega_{2/k} \\
\|\nabla \varphi_k\|_\infty \leq Ck
\end{cases}
$$

$$
\begin{cases}
w_{k,n} := \varphi_k u_n + (1-\varphi_k)u \\
w_{k,n} = u \quad \text{outside } \Omega_{1/k} \\
w_{k,n} = u_n \quad \text{in } \Omega_{2/k}
\end{cases}
$$

- $\overline{\lim_k}\ \overline{\lim_n}\ \|w_{k,n} - u\|_{L^\infty(\Omega;\mathbb{R}^d)} = 0$
- $\overline{\lim_k}\ \overline{\lim_n}\ \|\nabla w_{k,n}\|_{L^\infty} \leq \sup_n \|\nabla u_n\|_{L^\infty} + \|\nabla u\|_{L^\infty} < +\infty$

and
$$
\int_\Omega f(\nabla w_{k,n})\,dx \leq \int_\Omega f(\nabla u_n)\,dx + \underbrace{\int_{\Omega\setminus\Omega_{2/k}} f(\nabla u)\,dx}_{\to 0 \text{ as } k\to+\infty}
$$
$$
+ \underbrace{\int_{\Omega_{1/k}\setminus\Omega_{2/k}} f(\nabla w_{k,n})\,dx}_{\leq C|\Omega_{1/k}\setminus\Omega_{2/k}| \not\to 0}
$$

because $|\Omega|=+\infty$!

So need to fix $k \in \mathbb{N}$ and let the transition layer disappear before sending $k \to +\infty$:

let $0 < s << 1$ be such that $\frac{1}{k} > s > 0$ and choose

$$\begin{cases} \varphi_{k,s} \in C_c^\infty(\Omega_{2/k}; [0,1]) \\ \varphi_{k,s} \equiv 1 \text{ in } \Omega_{\frac{1}{k}+s} \\ \|\nabla \varphi_{k,s}\|_\infty \leq \frac{C}{s} \end{cases}$$

$$w_{k,s,n} := \varphi_{k,s} u_n + (1-\varphi_{k,s}) u$$

- $\overline{\lim_k} \overline{\lim_s} \overline{\lim_n} \| w_{k,s,n} - u \|_{L^\infty(\Omega; \mathbb{R}^d)} = 0$
- $\overline{\lim_k} \overline{\lim_s} \overline{\lim_n} \| \nabla w_{k,s,n} \|_{L^\infty(\Omega; \mathbb{R}^{d \times N})} < +\infty$
- $\int_\Omega f(\nabla w_{k,s,n}) dx \leq \int_\Omega f(\nabla u_n) dx + \int_{\Omega \setminus \Omega_{2/k}} f(\nabla u) dx$

$$+ \int_{\Omega_{2/k} \setminus \Omega_{1/k+s}} f(\nabla w_{k,s,n}) dx.$$

For fixed k, s choose $n(k,s)$ s.t. $n \geq n(k,s)$

$$\sup_n \| u_n - u \|_{L^\infty(\Omega_{2/k})} \quad \| \nabla \varphi_{k,s} \|_{L^\infty} \leq 1$$

$$C_0 := \sup_n \| \nabla u_n \|_{L^\infty} + \| \nabla u \|_{L^\infty} + 1$$

$$M := \sup \{ |f(\xi)| : |\xi| \leq C_0 \}$$

Then
$$\overline{\lim_{\substack{n\to+\infty \\ n\geq n(k,s)}}} \int_\Omega f(\nabla w_{k,s,n})\,dx \leq \lim_{n\to+\infty}\int_\Omega f(\nabla u_n)\,dx$$
$$+ \int_{\Omega\setminus\Omega_{2/k}} f(\nabla u)\,dx + \underbrace{M\,|\Omega_{1/k}\setminus\Omega_{1/k+s}|}_{\to\, 0 \text{ as } s\to 0^+}$$

and so
$$\underline{\lim}_{k\to+\infty}\ \overline{\lim}_{s\to 0^+}\ \overline{\lim}_{n\to+\infty} \int_\Omega f(\nabla w_{k,s,n})\,dx \leq \lim_{n\to+\infty}\int_\Omega f(\nabla u_n)\,dx.$$

Diagonalize to get $w_k = w_{k,\,s(k),\,n(k)}$.

Now prove:

Thm

$\Omega\subset\mathbb{R}^N$ open, $f:\mathbb{R}^{d\times N}\to[0,+\infty)$ lsc, locally bounded from above

Assume $\exists\, u_0\in V^{1,p}(\Omega;\mathbb{R}^d)$ s.t.
$$\int_\Omega f(\nabla u_0)\,dx\in\mathbb{R}.$$

If $u\in V^{1,p}(\Omega;\mathbb{R}^d)\mapsto \int_\Omega f(\nabla u)\,dx$ is swlsc in $V^{1,p}$ (w-* if $p=+\infty$) then f is strongly quasiconvex (hence, quasiconvex).

Recall :

Thm

$\Omega \subset \mathbb{R}^N$ open, bounded, $1 \le p \le +\infty$

$f: \mathbb{R}^{d\times N} \to (-\infty, +\infty]$ lsc.

(H) $\quad f(\xi) \ge -c(1+|\xi|^p)$ some $c>0$, if $1\le p<+\infty$

f locally bounded from below if $p=+\infty$.

If $\quad u \in W^{1,p}(\Omega;\mathbb{R}^d) \mapsto \int_\Omega f(\nabla u)\,dx$ is swlsc in $W^{1,p}$

(w-* if $p=+\infty$)

then f is strongly quasiconvex (hence, quasiconvex).

N.B. $\quad$ (H) + (strong) quasiconvexity are not sufficient to guarantee swlsc of the functional:

$N=2$, $\Omega := (-a,a)\times(0,a)$ with $0<a<1$

$$u_n(x_1,x_2) := \frac{1}{\sqrt{n}}(1-|x_2|)^m(\sin nx_1, \cos nx_1)$$

$f(\xi) := \det \xi \qquad$ polyconvex (hence strongly qcx)

and $|f(\xi)| \le c(1+|\xi|^2)$

Still, $\quad u_n \rightharpoonup 0$ in $W^{1,2}(\Omega;\mathbb{R}^2)$ with

$$\varliminf_{n\to+\infty} \int_\Omega f(\nabla u_n)\,dx < 0 = \int_\Omega f(0)\,dx.$$

The problem is the bound from below:

$f^-(\xi) \sim |\xi|^2$ and not better!