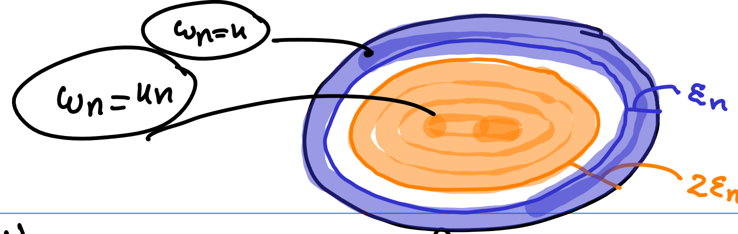

Lemma

$\Omega \subset \mathbb{R}^N$ open, $f : \mathbb{R}^{d\times N} \to [0,+\infty)$ Borel function, $C > 0$ s.t.

- if $|\Omega| < +\infty$ then $1 \leq p \leq +\infty$ and
  $0 \leq f(\xi) \leq C(1+|\xi|^p)$ for all $\xi \in \mathbb{R}^{d\times N}$, if $1 \leq p < +\infty$
  f is locally bounded if $p = +\infty$

- if $|\Omega| = +\infty$ then
  $0 \leq f(\xi) \leq C|\xi|^p$ for all $\xi \in \mathbb{R}^{d\times N}$, if $1 \leq p < +\infty$
  f is locally bounded if $p = +\infty$

Let $u \in V^{1,p}(\Omega;\mathbb{R}^d)$, $\{u_n\} \subset V^{1,p}(\Omega;\mathbb{R}^d)$ be such that

$$\lim_{n\to+\infty} \int_\Omega f(\nabla u_n)\,dx \text{ exists and is finite}$$

and $\int_\Omega f(\nabla u)\,dx \in \mathbb{R}$ if $p=+\infty$, $|\Omega| = +\infty$.

- $p=1$, $u_n \to u$ in $L^1_{loc}(\Omega;\mathbb{R}^d)$
- $1<p<+\infty$, $u_n \rightharpoonup u$ in $V^{1,p}(\Omega;\mathbb{R}^d)$
- $p=+\infty$, $u_n \overset{*}{\rightharpoonup} u$ in $V^{1,\infty}(\Omega;\mathbb{R}^d)$

Then there exists a sequence $\{w_n\} \subset V^{1,p}(\Omega;\mathbb{R}^d)$ such that

- $\|w_n - u\|_{L^p(\Omega;\mathbb{R}^d)} \to 0$, $1 \leq p \leq +\infty$
- $w_n = u$ on $\Omega \setminus \Omega_{\varepsilon_n}$, $w_n = u_n$ in $\Omega_{2\varepsilon_n}$, some $\varepsilon_n \searrow 0^+$
- $w_n \rightharpoonup u$ in $V^{1,p}(\Omega;\mathbb{R}^d)$ if $1<p<+\infty$
  $w_n \overset{*}{\rightharpoonup} u$ in $V^{1,\infty}(\Omega;\mathbb{R}^d)$ if $p=+\infty$

If, in addition, $u_n \rightharpoonup u$ in $V^{1,1}(\Omega;\mathbb{R}^d)$ then $w_n \rightharpoonup u$ in $V^{1,1}(\Omega;\mathbb{R}^d)$

- $$\boxed{\overline{\lim_{n\to+\infty}} \int_\Omega f(\nabla w_n)\,dx \leq \lim_{n\to+\infty} \int_\Omega f(\nabla u_n)\,dx}$$
- $$\boxed{\lim_{n\to+\infty} \left| \int_\Omega f(\nabla w_n)\,dx - \int_\Omega f(\nabla u_n)\,dx \right| = 0} \text{ if } p=+\infty,\ |\Omega| < +\infty$$