Corollary

$f: \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ Borel function

$\Omega \subset \mathbb{R}^N$ open, bounded, $|\partial\Omega| = 0$

For $r > 0$, define for $\xi \in \mathbb{R}^{d\times N}$

$$G_{\Omega,r}(\xi) := \inf\left\{ \fint_\Omega f(\xi + \nabla\varphi(x))\,dx : \varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^d), \; \|\xi + \nabla\varphi\|_{L^\infty(\Omega)} \le r \right\}$$

(whenever the integral is well defined)

(i) $G_{\Omega,r}$ is independent of the domain Ω;

(ii) $G_\Omega \le G_{\Omega,r} \le G_{\Omega,s}$ for $0 < s \le r$;

(iii) $G_{\Omega,r}(\xi) \xrightarrow[r\to+\infty]{} G_\Omega(\xi)$ for all $\xi \in \mathbb{R}^{d\times N}$.

Corollary

$f: \mathbb{R}^{d\times N} \to [-\infty, +\infty)$ Borel function.

For all $\xi \in \mathbb{R}^{d\times N}$ there exists a sequence $\{\varphi_n\} \subset W_0^{1,\infty}(Q;\mathbb{R}^d)$ such that

$$Qf(\xi) = \lim_{n\to+\infty} \int_Q f(\xi + \nabla\varphi_n(x))\,dx$$

and $\|\varphi_n\|_{L^\infty(Q;\mathbb{R}^d)} \to 0$ as $n \to +\infty$.

Proposition

$f: \mathbb{R}^{d\times N} \to [-\infty, +\infty)$ Borel function, $1 \le p < +\infty$.

For $\varepsilon > 0$ define $f_\varepsilon(\xi) := f(\xi) + \varepsilon|\xi|^p$, $\xi \in \mathbb{R}^{d\times N}$.

Then $\boxed{Qf_\varepsilon(\xi) \xrightarrow[\varepsilon\to 0^+]{} Qf(\xi)}$, $\forall\, \xi \in \mathbb{R}^{d\times N}$.

Morrey's Conjecture: the notion of quasiconvexity is nonlocal, i.e., there is no general condition involving only f and a finite number of derivatives of f that is both necessary and sufficient for quasiconvexity of f.

N.B. rank-1 convexity is local (Legendre-Hadamard cond.)

Definition

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ Borel function is said to be locally quasiconvex if $\forall\, \xi \in \mathbb{R}^{d\times N}$ $\exists\, r = r(\xi) > 0$ $\exists\, g : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ quasiconvex, s.t.

$$g(\eta) = f(\eta) \text{ for all } \eta \in B(\xi, r).$$

Kristensen proved that there are locally quasiconvex functions that are not quasiconvex, thus validating Morrey's conjecture.

back to swlsc of energy functionals...

Recall that we proved the following Thm:

Thm

$\Omega \subset \mathbb{R}^N$ open, bounded, $1 \leq p \leq +\infty$

$f : \mathbb{R}^{d\times N} \to (-\infty, +\infty]$ lsc

- $f(\xi) \geq -C(1+|\xi|^p)$ some $C>0$, all $\xi \in \mathbb{R}^{d\times N}$, if $p<+\infty$
- f locally bounded from below if $p=+\infty$

If $u \in W^{1,p}(\Omega;\mathbb{R}^d) \mapsto \int_\Omega f(\nabla u(x))dx$ is swlsc (w-* if $p=+\infty$)

then f is strongly quasiconvex (hence, quasiconvex)

Rmk The proof used the Riemann-Lebesgue Lemma:

$Y \subset \mathbb{R}^N$ unit cube, $\varphi \in W^{1,p}_{loc}(\mathbb{R}^N;\mathbb{R}^d)$, $\nabla\varphi$ Y-periodic

$f(\nabla\varphi(\cdot)) \in L^1(\Omega)$,

$$u_n(x) := \frac{1}{n}\varphi(nx).$$

Then

$$f(\nabla u_n) \rightharpoonup \int_Y f(\nabla\varphi(x))dx \quad \text{in } L^1_{loc}(\Omega). \quad (*)$$

If Ω is not bounded, even if $|\Omega|<+\infty$, $(*)$ may be false: there exist a Q-periodic function $v \in L^1_{loc}(\mathbb{R}^N)$, and an unbounded set Ω with $|\Omega|<+\infty$ s.t.

$$v \notin L^1(\Omega).$$

If we drop boundedness of Ω but keep $|\Omega| < +\infty$:

Thm

$\Omega \subset \mathbb{R}^N$ open, $|\Omega| < +\infty$

$f : \mathbb{R}^{d \times N} \to \mathbb{R}$ lsc, locally bounded from above.

- $f(\xi) \geq -C(1 + |\xi|^p)$ for some $C \geq 0$, all $\xi \in \mathbb{R}^{d \times N}$, if $p < +\infty$
- f locally bounded from below if $p = +\infty$.

If $u \in V^{1,p}(\Omega; \mathbb{R}^d) \mapsto \int_\Omega f(\nabla u(x))\,dx$ is swlsc (w-$*$ if $p = +\infty$) in $V^{1,p}(\Omega; \mathbb{R}^d)$ then

f is strongly quasiconvex (hence, quasiconvex)

Next, we consider the case in which $|\Omega| = +\infty$.

Thm

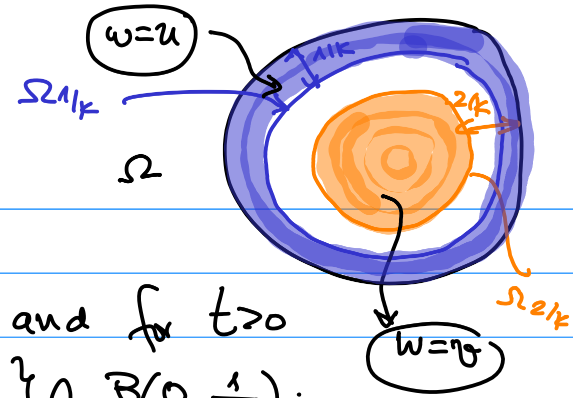
$\Omega \subset \mathbb{R}^N$ open, $f : \mathbb{R}^{d \times N} \to [0, +\infty)$ lsc, locally bounded from above

Assume $\exists\ u_0 \in V^{1,p}(\Omega; \mathbb{R}^d)$ s.t. $\int_\Omega f(\nabla u_0(x))\,dx \in \mathbb{R}$.

If $u \in V^{1,p}(\Omega; \mathbb{R}^d) \mapsto \int_\Omega f(\nabla u(x))\,dx$ is swlsc (w-$*$ if $p = +\infty$) in $V^{1,p}(\Omega; \mathbb{R}^d)$ then

f is strongly quasiconvex (hence, quasiconvex).

The proof uses De Giorgi's Slicing Lemma:



De Giorgi's Slicing Lemma

Let $\Omega \subset \mathbb{R}^n$ be an open set, and for $t>0$

$$\Omega_t := \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)>t\} \cap B\left(0,\tfrac{1}{t}\right).$$

Let $1\le p<+\infty$ and $f:\mathbb{R}^{d\times N}\to[0,+\infty]$ Borel function s.t. there exists $C>0$:

- $0 \le f(\xi) \le C(1+|\xi|^p)$ for all $\xi\in\mathbb{R}^{d\times N}$, if $|\Omega|<+\infty$
- $0 \le f(\xi) \le C|\xi|^p$ for all $\xi\in\mathbb{R}^{d\times N}$, if $|\Omega|=+\infty$.

Let $u,v\in V^{1,p}(\Omega;\mathbb{R}^d)$, $k,m\in\mathbb{N}$. Then there exists $\varphi\in C_c^\infty(\Omega_{1/k};[0,1])$ such that

$$\varphi=1 \text{ in } \Omega_{2/k}, \qquad \|\nabla\varphi\|_\infty \le Cmk$$

s.t. $\quad w := \varphi v + (1-\varphi)u$ satisfies

- $w=u$ on $\Omega\setminus\Omega_{1/k}$, $w=v$ in $\Omega_{2/k}$
- $\|w-u\|_{L^p(\Omega;\mathbb{R}^d)} \le \|v-u\|_{L^p(\Omega_{1/k};\mathbb{R}^d)}$

$$|\nabla w| \le |\nabla u| + |\nabla v| + Cmk\,\chi_{\Omega_{1/k}}|u-v| \quad \text{a.e. in } \Omega$$

- $$\int_\Omega f(\nabla w)\,dx \le \int_\Omega f(\nabla v)\,dx + C\int_{\Omega\setminus\Omega_{2/k}} \overbrace{(1+|\nabla u|^p)}^{(*)}\,dx + \frac{C}{m}\int_{\Omega_{1/k}\setminus\Omega_{2/k}} |\nabla v|^p\,dx + Ck^p m^{p-1}\int_{\Omega_{1/k}\setminus\Omega_{2/k}} |u-v|^p\,dx$$

if $|\Omega|<+\infty$, and remove "1" in (*) if $|\Omega|=+\infty$