Definition

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$

Rank-1 convex envelope of f, $Rf : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$, is the greatest rank-1 convex function below f.

Thm

$$\boxed{Rf(\xi) = \sup\{g(\xi) : g : \mathbb{R}^{d\times N} \to [-\infty,+\infty] \text{ rank-1 cx}, \ g \leq f\} \text{ for all } \xi \in \mathbb{R}^{d\times N}}$$

Another characterization of Rf:

Thm

let $R^1 f(\xi) := \inf\{\theta f(\xi + (1-\theta) a\otimes b) + (1-\theta) f(\xi - \theta a\otimes b) :$
$\theta \in [0,1],\ a \in \mathbb{R}^d,\ b \in \mathbb{R}^N$, and
$\{f(\xi + (1-\theta) a\otimes b), f(\xi - \theta a\otimes b)\} \neq \{-\infty, +\infty\}\}$.

Set $R^{n+1} f(\xi) := R^1(R^n f)(\xi)$, $\xi \in \mathbb{R}^{d\times N}$, $n \in \mathbb{N}$.

Then $R^{n+1} \leq R^n$, $n \in \mathbb{N}$, and

$$\boxed{Rf(\xi) = \lim_{n\to+\infty} (\inf) R^n f(\xi).}$$

Definition

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$

The quasiconvex envelope of f, $Qf : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$, is the greatest quasiconvex function below f, if it exists.

Rmk

let $\boxed{\tilde{Q}f(\xi) := \sup\{ g(\xi) : g: \mathbb{R}^{d\times N} \to [-\infty, +\infty] \text{ is quasiconvex}, \; g \le f \}.}$

1) If $\tilde{Q}f$ is a Borel function, then
$$\boxed{Qf = \tilde{Q}f.}$$

2) If $f: \mathbb{R}^{d\times N} \to [-\infty, +\infty)$ then Qf exists, and
either $Qf \equiv -\infty$
or $Qf : \mathbb{R}^{d\times N} \to \mathbb{R}$ is given by

$$\boxed{Qf(\xi) = \tilde{Q}f(\xi) = \sup\{ g(\xi) : g: \mathbb{R}^{d\times N} \to \mathbb{R} \text{ quasiconvex}, \; g \le f \}.}$$

Proposition

If $f: \mathbb{R}^{d\times N} \to [-\infty, +\infty)$ then $\boxed{Qf = Q(RF)}$

Thm

let $f: \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ be a Borel function, let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with $|\partial\Omega| = 0$.

Define for $\xi \in \mathbb{R}^{d\times N}$

$$\boxed{G_\Omega(\xi) := \inf\left\{ \fint_\Omega f(\xi + \nabla\varphi(x))\,dx : \varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^d) \text{ and the integral exists} \right\}.}$$

Then:

i) $G_\Omega \le f$ and if Qf exists then $Qf \le G_\Omega$;

ii) invariance of domain: if $A \subset \mathbb{R}^N$ is an open, bounded set with $|\partial A| = 0$, then $G_\Omega = G_A$;

iii) if $\xi \in \mathbb{R}^{d \times N}$ and $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^d)$ is piecewise affine then

$$G_\Omega(\xi) \leq \fint_\Omega G_\Omega(\xi + \nabla\varphi(x))\,dx$$

if the integral is well-defined;

iv) G_Ω is rank-1 convex at all $\xi \in \operatorname{int}(\operatorname{dom}_e f)$;

v) if $f : \mathbb{R}^{d\times N} \to [-\infty, +\infty)$ then

$$\boxed{Qf = G_\Omega.}$$