Definition

$f : \mathbb{R}^{d \times N} \to [-\infty, +\infty]$ Borel measurable, $1 \leq p \leq +\infty$, $\xi \in \mathbb{R}^{d \times N}$.

f is said to be $W^{1,p}$ - quasiconvex at ξ_0 if

$$f(\xi_0) \leq \int_Q f(\xi_0 + \nabla\varphi(x))\, dx$$

for all $\varphi \in W_0^{1,p}(Q; \mathbb{R}^d)$, whenever the integral exists.

Rmk

i) $W^{1,1}$ - quasiconvexity $\Rightarrow$ $W^{1,p}$ - quasiconvexity

$\Rightarrow \left[W^{1,\infty} \text{- quasiconvexity} = \text{quasiconvexity} \right]$

for all $1 < p < +\infty$.

2) $f : \mathbb{R}^{d \times N} \to [-\infty, +\infty)$ upper semi continuous

$f(\xi) \leq C(1 + |\xi|^p)$ for some $C > 0$, $1 \leq p < +\infty$.
all $\xi \in \mathbb{R}^{d \times N}$.

Then f is $W^{1,p}$ - quasiconvex at $\xi_0 \in \mathbb{R}^{d \times N}$ iff

$$f(\xi_0) \leq \int_\Omega f(\xi_0 + \nabla\varphi(x))\, dx$$

for some open, bounded $\Omega \subset \mathbb{R}^N$ and all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^d)$

Proposition

$f : \mathbb{R}^{d\times N} \to (-\infty, +\infty]$ Borel function.

(i) if $\quad f(\xi) \leq c(1+|\xi|^p) \quad$ some $c>0$, $1\leq p<+\infty$ all $\xi\in\mathbb{R}^{d\times N}$

then f is $W^{1,p}$-quasiconvex at $\xi\in\mathbb{R}^{d\times N}$ iff f is quasiconvex at ξ_0.

(ii) if $\quad f(\xi) \geq c|\xi|^p - \frac{1}{c} \quad$ some $c>0$, $1\leq p<+\infty$ all $\xi\in\mathbb{R}^{d\times N}$

then f is $W^{1,p}$-quasiconvex at $\xi_0\in\mathbb{R}^{d\times N}$ iff f is $W^{1,1}$-quasiconvex at ξ_0.

We already saw that $\xi\in\mathbb{R}^{N\times N} \mapsto \det\xi$ is a null-lagrangian, i.e.,

$$\det\xi = \int_Q \det(\xi+\nabla\varphi(x))\,dx$$

for all $\varphi\in W_0^{1,\infty}(Q;\mathbb{R}^d)$.

If $h: \mathbb{R}\to(-\infty,+\infty]$ is convex and lower semicontinuous, then Jensen's inequality yields

$$h(\det\xi) \leq \int_Q h(\det(\xi+\nabla\varphi(x)))\,dx$$

for all $\varphi\in W_0^{1,\infty}(Q;\mathbb{R}^N)$ for which RHS exists.

Therefore, $\xi \in \mathbb{R}^{N\times N} \longmapsto f(\xi) := h(\det \xi)$

is quasiconvex (actually, it belongs to the smaller class of polyconvex functions, i.e., convex functions of the minors of ξ).

" Conversely " ...

Proposition

$h: \mathbb{R} \longrightarrow [-\infty, +\infty]$ Borel measurable,

$f(\xi) := h(\det \xi)$, $\xi \in \mathbb{R}^{N\times N}$.

(i) h convex and lower semicontinuous $\Rightarrow$ f is $W^{1,p}$ quasiconvex, $N \leq p \leq +\infty$.

(ii) f quasiconvex $\Rightarrow$ h convex.

(iii) f $W^{1,p}$ quasiconvex, $1 \leq p < N$, iff h is constant.

Proof

Use radial functions $u(x) := g(|x|)\frac{x}{|x|}$, $x \neq 0$.

Then $\nabla u(x) = \frac{g(|x|)}{|x|}\mathbb{1} + \frac{g'(|x|)|x| - g(|x|)}{|x|}\frac{x}{|x|} \otimes \frac{x}{|x|}$,

and since $\det(\mathbb{1} + \eta \otimes \xi) = 1 + \eta\cdot\xi$,

get $\det \nabla u(x) = \left(\frac{g(|x|)}{|x|}\right)^{N-1} g'(|x|)$ ... ∎