Recall

Thm
$f: \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ quasiconvex at $\xi_0 \in \mathrm{int}(\mathrm{dom}_e f)$

$$\Downarrow$$

f rank-1 convex at ξ_0.

Rmk The proof of the above Thm does not use quasiconvexity in its full strength but only

$$f(\xi_0) \leq \int_Q f(\xi_0 + \nabla\varphi(x))\,dx$$

for $\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^d)$ piecewise affine (whenever the integral exist).

This will be used later on when we'll prove that G_Ω is rank-1 convex (re quasiconvex envelopes)

Rmk $f: \mathbb{R}^{d\times N} \to [-\infty, +\infty)$ quasiconvex, then either $f \equiv -\infty$ or $f: \mathbb{R}^{d\times N} \to \mathbb{R}$

Corollary $(d=1)$
$f: \mathbb{R}^{1\times N} \to [-\infty, +\infty]$ quasiconvex at $\xi_0 \in \mathrm{int}(\mathrm{dom}_e f)$

$$\Downarrow$$

f convex at ξ_0

Rmk The above Thm may fail if $\boxed{\xi_0 \notin \mathrm{int}(\mathrm{dom}_e f)}$

As an example, let $a \in \mathbb{R}^d$ and define $f : \mathbb{R}^{d\times N} \to (-\infty, +\infty]$ by

$$f(\xi) := \begin{cases} 0 & \text{if } \xi \in \{0, a\otimes e_N\}, \\ +\infty & \text{otherwise.} \end{cases}$$

Then f is lsc, it is not rank-1 convex:

$$+\infty = f(\theta\, a\otimes e_N) > \theta f(a\otimes e_N) + (1-\theta) f(0) = 0 \quad \forall \theta \in (0,1),$$

but still f is quasiconvex.

We introduce the notion of strongly quasiconvex, that is indeed the concept that arrives naturally as a necessary condition for swlsc in $V^{1,p}(\Omega;\mathbb{R}^d)$ of

$$u \in V^{1,p}(\Omega;\mathbb{R}^d) \mapsto \int_\Omega f(\nabla u(x))\,dx.$$

We'll see that even outside $\mathrm{int}(\mathrm{dom}_e f)$,

strongly quasiconvex $\Rightarrow$ rank-1 convex

Def

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ Borel function, is said to be **strongly quasiconvex** if

$$\boxed{\int_Y f(\xi + \nabla\varphi(x))\,dx \geq f\left(\xi + \int_Y \nabla\varphi(x)\,dx\right)}$$

for every unit cube $Y \subset \mathbb{R}^N$ and all $\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^N;\mathbb{R}^d)$ with $\nabla\varphi$ Y-periodic, whenever the left hand side is well-defined.

Proposition

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ strongly quasiconvex at $\xi_0 \in \mathbb{R}^{d\times N}$

$$\Downarrow$$

f is quasiconvex and rank-1 convex at ξ_0.

Rmk

Since there exist quasiconvex functions that are not rank-1 convex in view of the previous proposition

$$\text{strong quasiconvexity} \underset{\not\Leftarrow}{\Rightarrow} \text{quasiconvexity}$$

Proposition

If $f: \mathbb{R}^{d\times N} \to \mathbb{R}$ is quasiconvex then f is strongly quasiconvex

Rmk

For real-valued functions, $f: \mathbb{R}^{d\times N} \to \mathbb{R}$,

$$\boxed{\text{strong quasiconvexity} \Leftrightarrow \text{quasiconvexity}}$$