Last time:

$u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^d, \quad N, d \geq 1$

$u \in V^{1,p}(\Omega; \mathbb{R}^d) \longmapsto I(u) := \int_\Omega f(\nabla u(x))\,dx$ swlsc (w-* if $p = +\infty$)

$$\Downarrow$$

$$(*) \quad \boxed{\int_Y f(\xi + \nabla\varphi(x))\,dx \geq f\left(\xi + \int_Y \nabla\varphi(x)\,dx\right)}$$

for all cube $Y \subset \mathbb{R}^N$, $\varphi \in W^{1,\infty}_{loc}(Y; \mathbb{R}^d)$ with $\nabla\varphi$ Y-periodic

(whenever LHS exists)

If we restrit the testing in $(*)$ to $Y = Q := (-1/2, 1/2)^N$

and $\varphi \in W^{1,\infty}_0(Q; \mathbb{R}^d)$ (extended Q-periodically to $\mathbb{R}^N$)

then

$(*)$ becomes quasiconvexity:

$$(**) \quad \boxed{\int_Q f(\xi + \nabla\varphi(x))\,dx \geq f(\xi) \quad \text{for all } \varphi \in W^{1,\infty}_0(Q; \mathbb{R}^d) \quad \text{(whenever LHS exists)}}$$

So $\boxed{(*) \Rightarrow \text{quasiconvexity}}$

and later on we'll call $(*)$ ... strong quasiconvexity

$f : \mathbb{R}^{d \times N} \to (-\infty, +\infty]$ convex and lsc (automatic if $f < +\infty$)

then can apply Jensen's:

$$\boxed{\text{convex} + \text{lsc} \Rightarrow (*) \Rightarrow \text{quasiconvex}}$$

Example $\boxed{\begin{matrix}\text{quasiconvex} \\ \not\Downarrow \\ \text{convex}\end{matrix}}$ (and smooth and real-valued)

Indeed, $\det : \mathbb{R}^{N\times N} \longrightarrow \mathbb{R}^{N\times N}$, $N>1$,

$$\xi \longmapsto \det \xi$$

is a null Lagrangian, i.e.,

$$\int_Q \det(\xi + \nabla\varphi(x))\,dx = \det \xi$$

for all $\xi \in \mathbb{R}^{N\times N}$, $\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^d)$.

This + Jensen's $\Rightarrow$ $\xi \mapsto h(\det \xi)$ is quasiconvex

for all $h: \mathbb{R} \to \mathbb{R}$ convex function.

(actually, this is a polyconvex function).

In particular, $\xi \mapsto |\det \xi|$ is quasiconvex

but it is not convex! For $N=2$, $\theta \in (0,1)$

$$\left|\det\left[\theta\begin{pmatrix}1&0\\0&0\end{pmatrix} + (1-\theta)\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]\right| = \theta(1-\theta)$$

$$> 0 = \theta\left|\det\begin{pmatrix}1&0\\0&0\end{pmatrix}\right| + (1-\theta)\left|\det\begin{pmatrix}0&0\\0&1\end{pmatrix}\right|.$$

Intermediate between (*) and quasiconvexity
it is:

$$\boxed{(*-**)\quad \int_Q f(\xi + \nabla\varphi(x))\,dx \geq f(\xi) \quad \text{for all } \varphi \in W^{1,\infty}_{\#}(Q;\mathbb{R}^d)}$$

(whenever LHS exists)

where # stands for periodicity of φ.
We'll see that under appropriate growth conditions on f,

$$\boxed{(*-**) = (**) \ (\text{quasiconvexity})}$$

ie., can apply the quasiconvexity inequality to a larger class than $W^{1,\infty}_0(Q;\mathbb{R}^d)$.

The restriction in (**) (or (*-**)) to check the inequality only if $V = Q$, it is not really a restriction ... and this brings us to the invariance of the domain properties.

Proposition [Invariance of Domain]

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ quasiconvex at $\xi \in \mathbb{R}^{d\times N}$,

$\Omega \subset \mathbb{R}^N$ open, bounded, $|\partial\Omega| = 0$. Then

$$(M)_\Omega \quad \fint_\Omega f(\xi + \nabla\varphi(x))\,dx \geq f(\xi) \quad \text{for all } \xi \in W^{1,\infty}_0(\Omega;\mathbb{R}^d)$$

(whenever LHS exists)

Rmk 1) Using the same Morse's Covering Theorem argument, it can be shown that

$(M)_\Omega$ holds for Ω bounded, open (possible with $|\partial\Omega|>0$)

$$\Downarrow$$

$(M)_Q$ holds (for Q in place of Ω)
i.e., f is quasiconvex.

2) We will see next that if f is real-valued, then there is a different proof that does not require $|\partial\Omega|=0$ and yielding
M_Q (quasi convex) $\Rightarrow (M)_\Omega$

So, in view of 1), in this case
quasiconvexity $\Leftrightarrow (M)_\Omega$ for all $\Omega \subset \mathbb{R}^N$ open, bounded

... and the original definition of quasiconvexity of Morrey (who dealt with real-valued functions) is exactly that
$(M)_\Omega$ holds for all $\Omega \subset \mathbb{R}^N$ open, bounded.

Hence, the two "definitions" of quasiconvexity agree.

Proposition

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ quasiconvex at $\xi \in \mathbb{R}^{d\times N}$, $f(\xi) < +\infty$.

If $\Omega \subset \mathbb{R}^N$ is open, bounded, then

$$\fint_\Omega f(\xi + \nabla\varphi(x))\,dx \geq f(\xi) \quad \text{for all } \xi \in W_0^{1,\infty}(\Omega;\mathbb{R}^d).$$

(whenever LHS exists)

We already saw with the Example that, in general,

$$\text{quasiconvex} \not\Rightarrow \text{convex}$$
$$\Leftarrow$$

If $\min\{d, N\} = 1$ then the situation is different...

Theorem ($N=1$, $u : I \subset \mathbb{R} \to \mathbb{R}^d$)

$f : \mathbb{R}^d \to [-\infty, +\infty]$ quasiconvex at $\xi \in \mathbb{R}^d$

$$\Downarrow$$

f convex at ξ.

N.B. If $N=1$, then convexity and rank-1 convexity are the same, so above

$$f \text{ quasiconvex} \Rightarrow f \text{ rank-1 convex}$$

More generally:

Thm

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ quasiconvex

$\xi_0 \in \mathrm{int}(\mathrm{dom}_e f) \Rightarrow f$ rank-1 convex at ξ_0.

In particular

- f is rank-1 convex and locally Lipschitz in $\mathrm{int}(\mathrm{dom}_e f)$, and if $f < +\infty$ then either $f \equiv -\infty$ or f is real-valued and locally Lipschitz

- if, in addition,

$$|f(\xi)| \leq C(1+|\xi|^p) \quad \text{some } C>0,\ 1\leq p<+\infty, \text{ all } \xi \in \mathbb{R}^{d\times N}$$

then f is p-Lipschitz, i.e.,

$$|f(\xi_1)-f(\xi_2)| \leq C(1+|\xi_1|^{p-1}+|\xi_2|^{p-1})\,|\xi_1-\xi_2|$$

for all $\xi_1, \xi_2 \in \mathbb{R}^{d\times N}$

(the constants C vary from expression to expression)