Still about sufficiency, but now with $|\Omega| = +\infty$.

Thm

$\Omega \subset \mathbb{R}^N$ open, $|\Omega| = +\infty$

$f : \mathbb{R}^N \to (-\infty, +\infty]$ Borel measurable

f lsc and convex

$f(\xi) \geq 0$ for all $\xi \in \mathbb{R}^N$, if $1 < p \leq +\infty$

$f(\xi) \geq b.\xi$ for some $b \in \mathbb{R}^N$, all $\xi \in \mathbb{R}^N$, if $p = 1$

Then $u \in V^{1,p}(\Omega) \mapsto \int_\Omega f(\nabla u(x))\,dx$

is swlsc (w-* if $p = +\infty$) in $V^{1,p}(\Omega)$.

Rmk. Recall that, in view of last lecture, the hypotheses on f are not restrictive

Proof Uses the "blow-up" method. ■

In the vector-valued case, we have the necessary condition of f ... quasiconvexity!

Precisely, here $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^d$ and :

Thm

$\Omega \subset \mathbb{R}^N$ open, bounded, $1 \leq p \leq +\infty$

$f : \mathbb{R}^{d \times N} \to (-\infty, +\infty]$ lower semi continuous.

$$f(\xi) \geq -c(1+|\xi|^p) \quad \text{for some } c \in \mathbb{R}, \text{ all } \xi \in \mathbb{R}^{d\times N}$$
$$\text{if } 1 \leq p < +\infty$$

f locally bounded from below if $p = +\infty$.

If $u \in V^{1,p}(\Omega;\mathbb{R}^d) \mapsto \int_\Omega f(\nabla u(x))dx$

is swlsc in $V^{1,p}(\Omega;\mathbb{R}^d)$ (w-* if $p=+\infty$) then for all $\xi \in \mathbb{R}^{d\times N}$

$$\boxed{\int_Y f(\xi + \nabla\varphi(x))dx \geq f\left(\xi + \int_Y \nabla\varphi(x)\,dx\right)}$$

for all cube $Y \subset \mathbb{R}^N$ and all $\varphi \in W^{1,p}_{loc}(\mathbb{R}^N;\mathbb{R}^d)$

with $\nabla\varphi$ Y-periodic.

In particular, f is quasiconvex.

The proof uses the following corollary of the Riemann-Lebesgue lemma:

Corollary

Let $u \in W^{1,p}_{loc}(\mathbb{R}^N;\mathbb{R}^d)$, $1 \leq p \leq +\infty$, ∇u is Y-periodic,

with $Y \subset \mathbb{R}^N$ cube. Set

$$u_\varepsilon(x) := \varepsilon\, u\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0, \; x \in \mathbb{R}^N.$$

Then $u_\varepsilon \rightharpoonup u_0$ in $W^{1,p}_{loc}(\mathbb{R}^N;\mathbb{R}^d)$ (w-* if $p=+\infty$)

where $u_0(x) := \left(⨍_Y \nabla u(y)\,dy\right)x, \quad \forall x \in \mathbb{R}^N$

We now study the notion of Quasiconvexity

recall...

Definition

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ is said to be convex at $\xi \in \mathbb{R}^{d\times N}$ if

$$f(\xi) \leq \theta f(\xi + (1-\theta)\zeta) + (1-\theta) f(\xi - \theta\zeta)$$

for all $\theta \in (0,1)$, $\zeta \in \mathbb{R}^{d\times N}$, such that

$$\{ f(\xi + (1-\theta)\zeta), f(\xi - \theta\zeta) \} \neq \{-\infty, +\infty\}$$

f is convex if it is convex at all $\xi \in \mathbb{R}^{d\times N}$.

Definition

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ is said to be rank-1 convex at $\xi \in \mathbb{R}^{d\times N}$ if

$$\boxed{f(\xi) \leq \theta f(\xi + (1-\theta) a\otimes b) + (1-\theta) f(\xi - \theta a\otimes b)}$$

for all $\theta \in (0,1)$, $a \in \mathbb{R}^d$, $b \in \mathbb{R}^N$, such that

$$\{ f(\xi + (1-\theta) a\otimes b), f(\xi - \theta a\otimes b) \} \neq \{-\infty, +\infty\}$$

f is rank-1 convex if it is convex at all $\xi \in \mathbb{R}^{d\times N}$

Here, for $a \in \mathbb{R}^d$, $b \in \mathbb{R}^N$, $a\otimes b$ is the rank-1 matrix with entries

$$(a\otimes b)_{ij} := a_i b_j, \quad i \in \{1,\dots,d\}, \ j \in \{1,\dots,N\}.$$

Rmk (i) $\boxed{f \text{ convex} \Rightarrow f \text{ rank-1 convex}}$

(ii) f rank-1 convex at $\xi \in \mathbb{R}^{d\times N}$ $\Leftrightarrow$ $t \mapsto f(\xi + t\, a\otimes b)$ is convex

for all $a \in \mathbb{R}^d$, $b \in \mathbb{R}^N$.

In particular, if $f \in C^2$ then f is rank-1 convex at $\xi \in \mathbb{R}^{d\times N}$ iff

$$\frac{d^2 f}{dt^2}(\xi + t\, a\otimes b)\Big|_{t=0} \geq 0,$$

i.e., the Legendre-Hadamard condition holds:

$$\nabla^2 f(\xi)\, a\otimes b \cdot a\otimes b = \sum_{i,k=1}^{d} \sum_{j,l=1}^{N} \frac{\partial^2 f}{\partial \xi_{ij}\, \partial \xi_{kl}}(\xi)\, a_i\, b_j\, a_k\, b_l \geq 0$$

for all $a \in \mathbb{R}^d$, $b \in \mathbb{R}^N$.

(iii) $f : \mathbb{R}^{d\times N} \to (-\infty, +\infty]$ rank-1 convex

$\Rightarrow$ f is separately convex

$\Rightarrow$ $\boxed{f \text{ is locally Lipschitz in } \operatorname{int}(\operatorname{dom}_e f)}$

with effective domain of $f = \operatorname{dom}_e(f)$
$:= \{\xi \in \mathbb{R}^{d\times N} : f(\xi) < +\infty\}$.

(iv) $f : \mathbb{R}^{d\times N} \to [-\infty, +\infty)$ rank-1 convex.

Then either $f \equiv -\infty$ or $f : \mathbb{R}^{d\times N} \to \mathbb{R}$.

Definition

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ Borel function is said to be **quasiconvex at $\xi \in \mathbb{R}^{d\times N}$** if

$$\boxed{\int_Q f(\xi + \nabla\varphi(x))\,dx \geq f(\xi)}$$

for every $\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^d)$ for which LHS exists on $[-\infty, +\infty]$, and where $Q := (-1/2, 1/2)^N$.

f is **quasiconvex** if it is quasiconvex at all $\xi \in \mathbb{R}^{d\times N}$.
f is **quasi-affine** (or **null lagrangian**) if $\pm f$ quasiconvex.

Rmk

$$\boxed{\begin{array}{c} f : \mathbb{R}^{d\times N} \to (-\infty, +\infty] \text{ lsc + convex} \\ \Downarrow \\ f \text{ is quasiconvex.} \end{array}}$$

It suffices to use **Jensen's Inequality**:

$$\boxed{\begin{array}{l} X \text{ Banach space, } f : X \to (-\infty, +\infty] \text{ lsc + convex} \\ \mu \text{ probability measure on } (X, \mathcal{M}) \\ g \in L^1(X, \mathcal{M}, \mu). \\ \\ \text{Then } \int_X f\circ g \, d\mu \geq f\left(\int_X g\, d\mu\right). \end{array}}$$

We will see that quasiconvex $\not\Rightarrow$ convex