Decomposition Lemma in Sobolev Spaces

$\Omega \subset \mathbb{R}^N$ open set, $1 < p < +\infty$

$\{u_n\} \subset V^{1,p}(\Omega;\mathbb{R}^d)$, $u_n \rightharpoonup u_0$ in $V^{1,p}(\Omega;\mathbb{R}^d)$.

There exist a subsequence $\{u_{n_i}\} \subset \{u_n\}$ and $\{w_i\} \subset W^{1,p}(\Omega;\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ s.t.

- $w_i \rightharpoonup 0$ in $W^{1,p}(\Omega;\mathbb{R}^d)$;
- $\{|\nabla w_i|^p\}$ is equi-integrable;
- $\forall R > 0$ $\quad |\{x \in \Omega \cap B(0,R) : u_0(x) + w_i(x) \neq u_{n_i}(x)\}| \xrightarrow[i \to +\infty]{} 0$.

Moreover, if $|\Omega| < +\infty$ then $\operatorname{supp} w_i \subset\subset \Omega$.

Rmk Can decompose

$$\boxed{u_{n_i} = \underbrace{(u_0 + w_i)}_{z_i} + v_i}$$

- $z_i \rightharpoonup u_0$ in $V^{1,p}$;
- $\{|\nabla z_i|^p\}$ is equi-integrable;
- $|\{x \in \Omega \cap B(0,R) : v_i(x) \neq 0\}| \xrightarrow[i]{} 0$, $\forall R > 0$.

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Next ... back to swlsc

$$u \in V^{1,p} \longmapsto I(u) := \int_\Omega f(\nabla u)\,dx$$

but now $\boxed{|\Omega| = +\infty}$

Recall

$|\Omega| < +\infty$, $f: \mathbb{R}^N \to (-\infty, +\infty]$ Borel measurable

If $u \mapsto \int_\Omega f(\nabla u)\,dx$

is swlsc in $W^{1,p}$ whenever the target is affine
then f is convex

N.B. If $|\Omega| = +\infty$ then affine functions $\notin V^{1,p}$,
so need a different strategy...

To prove next time:

Thm $|\Omega| = +\infty$, $f: \mathbb{R}^N \to \mathbb{R}$ Borel measurable
and such that

- $0 \leq f(\xi) \leq C(1 + |\xi|^p)$ if $1 \leq p < +\infty$;
- $0 \leq f$ and f is locally bounded from above, $p = +\infty$

Suppose that $\exists\, v_0 \in V^{1,p}(\Omega)$ s.t. $\int_{\mathbb{R}} f(\nabla v_0)\,dx \in \mathbb{R}$.

If $u \mapsto \int_\Omega f(\nabla u)\,dx$

is swlsc in $V^{1,p}$ ($\overset{*}{\rightharpoonup}$ if $p = +\infty$) then f is convex.

Rmk: 1. If $v_0 \in W^{1,p}$ then it suffices to have swlsc
($\overset{*}{\rightharpoonup}$ if $p = +\infty$) in $W^{1,p}$.

2. The constraint that $\exists\, v_0 \in V^{1,p}$ s.t.

$$\int_\Omega f(\nabla v_0)\,dx \in \mathbb{R}$$

is "not restrictive", since otherwise $I \equiv +\infty$
(not interesting!).

3. If $\boxed{1 < p \leq +\infty}$, then the condition $f \geq 0$
is "not restrictive". Indeed, we will show that

- $|\Omega| = +\infty$, f lsc + convex, $I(\cdot)$ is well posed, i.e.
$$f(\xi) \geq -c|\xi|^p \quad , \quad 1 \leq p < +\infty$$

(if $p = +\infty$ then well posedness $\Rightarrow$ $f \geq 0$)

- and $\exists\, v_0 \in V^{1,p}$ s.t. $\int_{\mathbb{R}} f(\nabla v_0) \in \mathbb{R}$

$$\Downarrow$$

$$\boxed{f(0) = 0 \quad \text{and} \quad f(\xi) \geq 0 \quad \text{for all } \xi.}$$

If $\boxed{p = 1}$, we can only guarantee that
$$\boxed{f(\xi) \geq b \cdot \xi \quad \forall\, \xi \in \mathbb{R}^N, \text{ some } b \in \mathbb{R}^N.}$$