We will need the notion of equi-integrability:

Def $(X, \mathcal{M}, \mu)$ measure space, $\mathcal{F}$ family of measurable functions $u: X \to [-\infty, +\infty]$.

(i) $\mathcal{F}$ is equi-integrable if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.
$$\sup_{u \in \mathcal{F}} \int_E |u| \, d\mu < \varepsilon$$
for every $E \in \mathcal{M}$ with $\mu(E) < \delta$.

(ii) $\mathcal{F}$ is p-equi-integrable, $1 \leq p < +\infty$, if $\{ |u|^p : u \in \mathcal{F} \}$ is equi-integrable.

Recall: $1 < p \leq +\infty$, $\sup_n \|u_n\|_{L^p} < +\infty \Rightarrow \exists$ subsequence

$\{u_{n_k}\} \subset \{u_n\}$ s.t. $u_{n_k} \rightharpoonup u$ in L^p ($\overset{*}{\rightharpoonup}$ if $p = +\infty$) for some $u \in L^p$.

This is, in general, false for $p = 1$. We need more than just L^1 bounds ...

Thm [Dunford-Pettis]

$(X, \mathcal{M}, \mu)$ measure space, $\mathcal{F} \subset L^1(X)$.

$\mathcal{F}$ is sequentially weakly pre-compact iff

(i) $\mathcal{F}$ is bounded in $L^1(X)$;

(ii) $\mathcal{F}$ is equi-integrable, and if $\mu(X) = +\infty$ then $\forall \varepsilon > 0 \ \exists E \in \mathcal{M}$ s.t. $\mu(E) < +\infty$ and
$$\sup_{u \in \mathcal{F}} \int_{X \setminus E} |u| \, d\mu < \varepsilon.$$

Proof See Fonseca & Leoni, Springer 2007, Thm 2.54. ∎

Decomposition Lemma in L^1

$(X, \mathfrak{M}, \mu)$ measure space, $u_n : X \to [-\infty, +\infty]$ measurable, $\sup_n \|u_n\|_{L^1(X)} < +\infty$.

$\forall\, r > 0$ consider the truncation map $\tau_r : \mathbb{R} \to \mathbb{R}$ defined as

$$\tau_r(z) := \begin{cases} z & \text{if } |z| \leq r, \\ \frac{z}{|z|} r & \text{if } |z| > r. \end{cases}$$

Then there exist a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and an increasing sequence of positive integers $j_k \nearrow +\infty$ s.t. $\{\tau_{j_k} \circ u_{n_k}\}$ is equi-integrable and

$$\mu(\{x \in X : u_{n_k}(x) \neq (\tau_{j_k} \circ u_{n_k})(x)\}) \xrightarrow[k \to \infty]{} 0.$$

Proof See Fonseca & Leoni, Springer 2007, Lemma 2.31. ∎

Rmk 1) We can decompose

$$\boxed{u_{n_k} = w_k + v_k}$$

where
- $\{w_k := \tau_{j_k} \circ u_{n_k}\}$ is equi-integrable
- $\mu(\{v_k \neq 0\}) \to 0$

2) If $u_n := |\nabla v_n|^p$, with $v_n \rightharpoonup v$ in $W^{1,p}$ for some $1 < p < +\infty$, applying the Decomposition Lemma directly on u_n would destroy the underlying curl $= 0$ constraint, i.e.,

$$u_{n_k} = w_k + v_k$$

but in general w_k cannot be written as $|\nabla z_k|^p$ for some $z_k \in W^{1,p}$.

Instead, the Decomposition Lemma will be applied to the sequence of Maximal Functions of $\{ |v_n|^p + |\nabla v_n|^p \}$.

REMARKS ON MAXIMAL FUNCTIONS

$\Omega \subset \mathbb{R}^N$ open, $u \in L^1_{loc}(\Omega)$, $E \subset \Omega$ measurable, $|E| < +\infty$. Then

$$u_E := \fint_E u(x)\,dx = \frac{1}{|E|} \int_E u(x)\,dx.$$

Def The Hardy-Littlewood Maximal Function of u is given by

$$\boxed{M(u)(x) := \sup_{r>0} \fint_{B(x,r)} |u(y)|\,dy.}$$

Thm 1 $u \in L^p(\mathbb{R}^N)$, $1 \leqq p \leqq +\infty$.

(i) $u(x) \leqq M(u)(x) < +\infty$ $\mathcal{L}^N$ a.e. $x \in \mathbb{R}^N$;

(ii) if $1 \leqq p < +\infty$ and $t > 0$ the

$$|\{x \in \mathbb{R}^N : M(u)(x) > t\}| \leqq \frac{3^N 2^p}{t^p} \int_{\{|u| > t/2\}} |u(y)|^p \, dy;$$

(iii) if $1 < p \leqq +\infty$ then $M(u) \in L^p(\mathbb{R}^N)$ and

$$\| M(u) \|_{L^p(\mathbb{R}^N)} \leqq C(N,p) \, \|u\|_{L^p(\mathbb{R}^N)}.$$

Proof See Fonseca & Leoni, Springer 2007, Thm 2.91. ∎

Thm 2 $u \in W^{1,p}_{loc}(\mathbb{R}^N)$, $1 \leqq p < +\infty$. Then

$$|u(x) - u(y)| \leqq C(N) \, |x-y| \, [M(|\nabla u|)(x) + M(|\nabla u|)(y)]$$

for $\mathcal{L}^N$ a.e. $x, y \in \mathbb{R}^N$.

Thm 3 $u \in V^{1,p}(\mathbb{R}^N)$, $1 \leqq p < +\infty$. $\forall t > 0$ there exists a Lipschitz function $v : \mathbb{R}^N \to \mathbb{R}$ such that

- $\operatorname{Lip} v \leqq C(N)\, t$;
- $u = v$ $\mathcal{L}^N$ a.e. on $\mathbb{R}^N \setminus A_t$, where

$$A_t := \{x \in \mathbb{R}^N : M(|\nabla u|)(x) > t\};$$

- $$|A_t| \leq \frac{C(N,p)}{t^p} \int_{\{|\nabla u| > t/2\}} |\nabla u|^p \, dx \; ;$$

- $$\|\nabla u - \nabla v\|^p_{L^p(\mathbb{R}^N;\mathbb{R})} \leq C(N,p) \int_{\{|\nabla u| > t/2\}} |\nabla u|^p \, dx .$$

Thm 4 $u \in W^{1,p}(\mathbb{R}^N)$, $1 \leq p < +\infty$. $\forall\, t > 0$ there exists a function $v \in W^{1,p}(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ s.t.

- $$\|v\|_{W^{1,\infty}(\mathbb{R}^N)} \leq C(N)\, t \; ;$$

- $u = v$ $\mathcal{L}^N$ a.e. on $\mathbb{R}^N \setminus D_t$, where

$$D_t := \{x \in \mathbb{R}^N : M(|u| + |\nabla u|)(x) > t\};$$

- $$|D_t| \leq \frac{C(N,p)}{t^p} \int_{\{|u| + |\nabla u| > t/2\}} (|u|^p + |\nabla u|^p) \, dx \, ;$$

- $$\|u - v\|^p_{W^{1,p}(\mathbb{R}^N)} \leq C(N,p) \int_{\{|u| + |\nabla u| > t/2\}} (|u|^p + |\nabla u|^p) \, dx .$$