s w lsc of $u \in V^{1,p}(\Omega) \mapsto I(u) = \int_\Omega f(\nabla u)\,dx$

$u_n \rightharpoonup u$ in $V^{1,p}$ $\Rightarrow$ $I(u) \leq \underline{\lim}\, I(u_n)$.

scalar case ($u : \Omega \to \mathbb{R}$)

Thm

$\Omega \subset \mathbb{R}^N$ open $|\Omega| < +\infty$, $1 \leq p \leq +\infty$

$f : \mathbb{R}^N \to (-\infty, +\infty]$, lsc

- $f(\xi) \geq -C(1 + |\xi|^p)$ $\quad 1 \leq p < +\infty$, $C > 0$
- f locally bounded from below, $p = +\infty$

$I(\cdot)$ is swlsc in $V^{1,p}$ (sw-* lsc, $p = +\infty$)
iff f is convex.

Rmk 1. Nec. of f being convex still holds if assume "only" swlsc of $I(\cdot)|_{W^{1,p}}$

2. Sufficiency of f being convex still holds $|\Omega| = +\infty$ and also in the vector-valued case, i.e. $V^{1,p}(\Omega) \mapsto V^{1,p}(\Omega; \mathbb{R}^d)$, $u : \Omega \to \mathbb{R}^d$

False for nec.

3. scalar valued case but $|\Omega| = +\infty$, nec. of f convex still holds with additional hypotheses.

Definition

$u : \mathbb{R}^N \to \mathbb{R}^d$ Q-periodic if

$$u(x+e_i) = u(x) \quad \text{for all } x \in \mathbb{R}^N,\ i = 1, \dots, N$$

$(e_1, \dots, e_N)$ cannonical basis of $\mathbb{R}^N$

More generally, u is kQ-periodic, $k \in \mathbb{N}$, if $u(k\cdot)$ is Q-periodic

$$u(k\cdot) \downarrow \quad x \mapsto u(kx)$$

Riemann-Lebesgue Lemma:

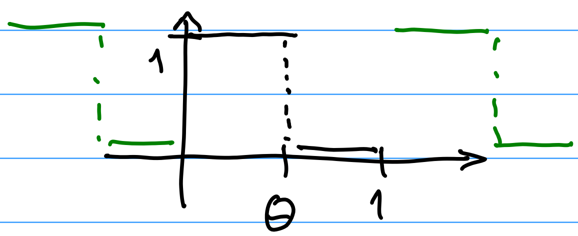
$u \in L^p_{loc}(\mathbb{R}^N)$, $1 \leq p \leq +\infty$, u kQ-periodic.

$\varepsilon > 0$ $\quad u_\varepsilon(x) := u\left(\frac{x}{\varepsilon}\right)$.

Then $u_\varepsilon \rightharpoonup \bar{u}$ in $L^p_{loc}(\mathbb{R}^N)$ ($\overset{*}{\rightharpoonup}$ if $p = +\infty$)

where $\bar{u} \equiv \fint_{kQ} u(y)\,dy = \frac{1}{k^N} \int_{Q(0,k)} u(y)\,dy.$

Example



$\theta \in (0,1)$

χ 1-periodic

$$u_\varepsilon(x) := \chi\left(\frac{x}{\varepsilon}\right)$$

$u_\varepsilon \overset{*}{\rightharpoonup} \bar{u}$ in L^∞_{loc}, $\bar{u} \equiv \int_0^1 \chi(y)\,dy = \theta$

$$\nu \in S^{N-1} := \{ x \in \mathbb{R}^N : |x| = 1 \}$$

$$u_\varepsilon(x) := \chi\left(\frac{x \cdot \nu}{\varepsilon}\right), \quad x \in \mathbb{R}^N$$

$$u_\varepsilon \xrightarrow{*} \bar{u} \text{ in } L^\infty_{loc}(\mathbb{R}^N), \quad \bar{u} \equiv \theta$$