Still on well-posedness ...

Corollary

$\Omega \subset \mathbb{R}^N$ open, $|\Omega| = +\infty$, $1 \leq p \leq +\infty$

$f: \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ Borel function

(*) $\int_\Omega f(\nabla u)\,dx$ is well defined for all $u \in V^{1,p}(\Omega;\mathbb{R}^d)$,

i.e., $\int_\Omega (f(\nabla u))^-\,dx$ and $\int_\Omega (f(\nabla u))^+\,dx$ are not both $+\infty$,

iff f or $-f$ (or both) satisfy

(i) $f(\xi) \geq -c|\xi|^p$ some $c>0$, all $\xi \in \mathbb{R}^{d\times N}$, if $1 \leq p < +\infty$

(ii) $f(\xi) \geq 0$ for all $\xi \in \mathbb{R}^{d\times N}$, if $p = +\infty$.

Rmk

The necessity of (i) + (ii) still holds if (*) is satisfied "only" for $u \in W_0^{1,p}(\Omega;\mathbb{R}^d)$

SEQUENTIAL LSC WITH RESPECT TO $V^{1,p}$ – strong

Thm

$\Omega \subset \mathbb{R}^N$ open, $|\Omega| < +\infty$, $1 \leq p \leq +\infty$

$f : \mathbb{R}^{d \times N} \to (-\infty, +\infty]$ Borel function s.t.

(iii) $f(\xi) \geq -C(1+|\xi|^p)$ some $C>0$, all $\xi \in \mathbb{R}^{d \times N}$ if $1 \leq p < +\infty$

(iv) f locally bounded from below if $p = +\infty$

The functional $u \in V^{1,p}(\Omega; \mathbb{R}^d) \mapsto I(u) := \int_\Omega f(\nabla u)\,dx$ is slsc with respect to strong convergence in $V^{1,p}$ iff f is lsc.

Rmk

The necessity of f being lsc still holds if we assume only slsc of $I|_{W^{1,p}}$.

Thm

$\Omega \subset \mathbb{R}^N$ open, $|\Omega| = +\infty$, $1 \leq p \leq +\infty$

$f : \mathbb{R}^{d \times N} \to (-\infty, +\infty]$ Borel function satisfying (i), (ii).

f lsc $\Rightarrow$ $I(\cdot)$ is slsc w.r. $V^{1,p}$ – strong convergence.

In the case in which $|\Omega| = +\infty$, we need additional assumptions to obtain lsc of f as a necessary condition.

Indeed:

Thm

$\Omega \subset \mathbb{R}^N$ open, $|\Omega| = +\infty$, $1 \leq p \leq +\infty$

$f : \mathbb{R}^{d\times N} \to \mathbb{R}$ Borel function s.t.

- $-c\,|\xi|^p \leq f(\xi) \leq c(1+|\xi|^p)$ some $c>0$, all $\xi \in \mathbb{R}^{d\times N}$ if $1 \leq p < +\infty$
- $f \geq 0$ and f is locally bounded from above if $p = +\infty$.

If there exists $u_0 \in V^{1,p}(\Omega; \mathbb{R}^d)$ s.t.

$$\int_\Omega f(\nabla u_0)\,dx \in \mathbb{R}$$

and if $I(\cdot)$ is slsc w.r. $V^{1,p}$ – strong convergence then f is lsc.

The proof uses the following lemma:

Lemma

$\Omega \subset \mathbb{R}^N$ open, $|\Omega| < +\infty$, $1 \leq p \leq +\infty$

$f : \mathbb{R}^{d\times N} \to \mathbb{R}$ Borel function s.t.

- $|f(\xi)| \leq c\,(1+|\xi|^p)$ for some $c>0$, all $\xi \in \mathbb{R}^{d\times N}$ if $1 \leq p < +\infty$
- f locally bounded if $p = +\infty$.

Let $\{u_n\} \subset V^{1,p}(\Omega; \mathbb{R}^d)$ be s.t. $u_n \to u$ in $V^{1,p}(\Omega;\mathbb{R}^d)$.

Then there exist $\varepsilon_n \to 0^+$ and $\{v_m\} \subset V^{1,p}(\Omega;\mathbb{R}^d)$ s.t.

- $v_m \to u$ in $V^{1,p}$;
- $v_m = u$ in a neighborhood of $\partial\Omega$;
- $v_m = u_n$ in $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon_n\}$;
- $$\lim_{n\to+\infty} \left| \int_\Omega f(\nabla u_n)\,dx - \int_\Omega f(\nabla v_m)\,dx \right| = 0 .$$