Thm $\Omega \subset \mathbb{R}^N$ open, $|\Omega| < +\infty$, $1 \leq p \leq +\infty$

$f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ Borel

Then $(*)$ $\displaystyle\int_\Omega (f(\nabla u))^- \, dx < +\infty \quad \forall u \in V^{1,p}(\Omega; \mathbb{R}^d)$.

iff (1). $\boxed{f(\xi) \geq -c(|\xi|^p + 1)}$, some $c \in \mathbb{R}$, all $\xi \in \mathbb{R}^{d\times N}$, $1 \leq p < +\infty$

(2). $\boxed{f \text{ locally bounded from below}}$, $p = +\infty$

1st proof + f usc

2nd proof general f, not nec. usc.

Rmk Nec. of (1) + (2) still holds if $(*)$ satisfied "only" in $W^{1,p}_0(\Omega; \mathbb{R}^d)$ and also if $|\Omega| = +\infty$.

As we will below, if $|\Omega| = +\infty$ then it is possible to obtain better lower bounds.

Rmk Applying the previous thm to f and $-f$, it follows that if $f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ is Borel the $f \circ \nabla u \in L^1(\Omega)$ $\forall u \in V^{1,p}$ iff

- $|f(\xi)| \leq c(|\xi|^p + 1)$ some $c > 0$, all $\xi \in \mathbb{R}^{d\times N}$ if $1 \leq p < +\infty$
- f is locally bounded if $p = +\infty$

Corollary $\Omega \subset \mathbb{R}^N$ open, $|\Omega| < +\infty$
$f: \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ Borel function.
Then $\int_\Omega f(\nabla u)\,dx$ is well defined for all $u \in V^{1,p}$

(i.e. $\int_\Omega (f(\nabla u))^- dx$ and $\int_\Omega (f(\nabla u))^+ dx$ cannot

be both $+\infty$) iff f or $-f$ (or both)
satisfy (1) if $1 \leq p < +\infty$, (2) if $p = +\infty$.

Thm $\Omega \subset \mathbb{R}^N$ open, $|\Omega| = +\infty$
$f: \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ Borel function
Then $\int_\Omega (f(\nabla u))^- dx < +\infty$ for all $u \in V^{1,p}$

iff
- $f(\xi) \geq -c|\xi|^p$ some $c \in \mathbb{R}$, all $\xi \in \mathbb{R}^{d\times N}$ if $1 \leq p < +\infty$
- $f \geq 0$ if $p = +\infty$.

The proof uses the Lemma:

Lemma $\Omega \subset \mathbb{R}^N$ open, $|\Omega| = +\infty$, $a > 0$.
Then there exists a finite family of open, mutually disjoint cubes $Q_i \subset \Omega$, $i = 1, \dots, n$, such that
$$\sum_{i=1}^n |Q_i| = a.$$