www.math.cmu.edu/PIRE/activities/courses.html

last time

$$\min\{I(u) : u \in A\}$$

$$\boxed{I(u) = \int_\Omega f(\nabla u(x))\,dx}$$

$u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^d$, $N, d \geq 1$

Direct Method of the Calc. Variations:

Q1 : necessary and/or ~~sufficiency~~ conditions on f that guarantee swlsc of $I(\cdot)$,

$$u_n \rightharpoonup u \Rightarrow I(u) \leq \liminf_{n\to+\infty} I(u_n)$$

Q2 : If $u \mapsto I(u)$ is NOT swlsc, then how can we characterize the "effective" energy of the system?

Q1 : we stated a thm with nec & suff for swlsc in L^p in the unconstrained case

$v \mapsto I(v)$, $I(v) := \int_E f(v(x))\,dx$

v not necessarily curl-free

$E \subset \mathbb{R}^N$ Lebesgue meas.

- sufficient conditions still hold:

if thm applies to $v_n \xrightarrow{LP} v$

also applies if in addition $\boxed{\operatorname{curl} v_n = 0}$

- If $E = \Omega$ open, connected set, $|\Omega| < +\infty$

then nec. conditions still hold if $\boxed{d=1}$

Recall Thm $|\Omega| < +\infty$, open connected

$1 \leq p \leq +\infty$, $f : \mathbb{R}^N \to (-\infty, +\infty]$ lsc

lower bounds
- $f(\xi) \geq -C(|\xi|^p + 1)$, if $1 \leq p < +\infty$
- f locally bounded from below, if $p = +\infty$

Then $u \mapsto I(u)$ is swlsc in $W^{1,p}$ ($\xrightarrow{*}$ $p = +\infty$)

IFF f is convex

Rmk lsc is not a restriction: if want $I(\cdot)$ swlsc, in particular needs to be lsc:

(*) $u_n \rightharpoonup u$ $W^{1,p}$ $\Rightarrow$ $I(u) \leq \underline{\lim}\, I(u_n)$

$\xi_n \in \mathbb{R}^N$, $\xi_n \to \xi$, $\xi \in \mathbb{R}^N$, define

$u_n(x) := \xi_n \cdot x$, $n \in \mathbb{N}$

$u(x) := \xi \cdot x$

$u_n \to u$ in $W^{1,p}$

(*) $\Rightarrow$ $\cancel{|\Omega|}\, f(\xi) = I(u) \leq \underline{\lim}\ I(u_n) = \underline{\lim}\ \cancel{|\Omega|}\, f(\xi_n)$

Rmk If $|\Omega| = +\infty$ the swlsc of $I(\cdot)$
$\not\Rightarrow$ f is convex ($d=1$)
(in contrast with the unconstrained case)

$\Omega = \mathbb{R}^2$, $e := (1,0)$

$$f(\xi) := \begin{cases} 0 & \text{if } \xi = 0 \\ 2 & \text{if } \xi = e \\ 1 & \text{if } \xi = \left(1+\frac{1}{n}\right)e, \ n \in \mathbb{N} \\ +\infty & \text{—} \end{cases}$$

$$\boxed{f \text{ not lsc } + \text{ not convex}}$$

still $u_n \rightharpoonup u$ $W^{1,p}$ $\Rightarrow \int_{\mathbb{R}^2} f(\nabla u) \leq \underline{\lim} \int_{\mathbb{R}^2} f(\nabla u_n)$

if $v \in W^{1,p}$: $\int_\Omega f(\nabla v)\,dx < +\infty \Rightarrow$ • $\nabla v \parallel e$ a.e.
• $|\{\nabla v \neq 0\}| < +\infty$

$\Rightarrow$ $\nabla v = 0$ a.e. $\Rightarrow$ $\int_{\mathbb{R}^2} f(\nabla u) = 0$

Q2 If $u \mapsto I(u)$ is not swlsc

It may happen $u_n \rightharpoonup u$ s.t.

$$\underline{\lim}\, I(u_n) < I(u)$$

$$\downarrow$$

relaxed energy

$$\boxed{\mathcal{F}(u) := \inf_{\{u_n\}} \left\{ \underline{\lim}\, I(u_n) : u_n \rightharpoonup u \right\}}$$

Can we find an integral representation for $\mathcal{F}$, i.e.,

$$\mathcal{F}(u) = \int_\Omega \bar{f}(x, u(x), \nabla u(x))\, dx$$

$\bar{f}$ ... effective or relaxed energy density

$$\inf \{ I(u) : u \in A \} = \text{"min"} \{ \mathcal{F}(u) : u \in A \}$$

In addition to failure of swlsc, need to deal with features not covered by "classical" Calc Var.:

- energies of $\neq$ dimensionality
- multiscale, singular perturbations
- higher order derivatives
- discontinuous underlying fields u

Sobolev setting $\to$ BV, ...

Concentrate first on swlsc of

$$\boxed{u \mapsto I(u) = \int_\Omega f(\nabla u(x))\,dx}$$

Morrey 60's : $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^d$, $N, d \geq 1$

swlsc of $I(\cdot)$ in $W^{1,p}$ $\sim$ $\boxed{\text{quasiconvexity}}$

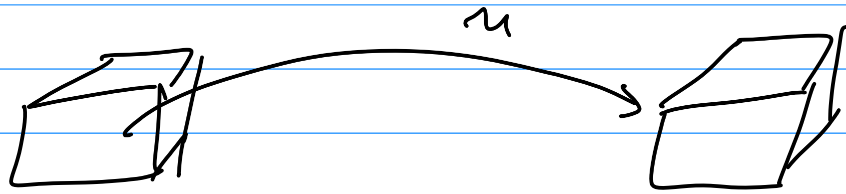
Def: $f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ Borel measurable

f is quasiconvex at $\xi_0 \in \mathbb{R}^{d\times N}$ if

$$\boxed{f(\xi_0) \leq \int_{(0,1)^N} f(\xi_0 + \nabla\varphi(x))\,dx \quad \forall\, \varphi \in W_0^{1,\infty}((0,1)^N; \mathbb{R}^d)}$$

whenever the RHS is well defined, i.e., cannot have both

$$\begin{cases} \int_{(0,1)^N} f^+(\xi_0+\nabla\varphi(x))\,dx = +\infty \\ \int_{(0,1)^N} f^-(\xi_0+\nabla\varphi(x))\,dx = +\infty \end{cases}$$



ref. conf.

$\begin{array}{|l} u_0(x) = \xi_0 x \\ u(x) = \xi_0 x + \varphi(x) \end{array}$

(*) **Convexity $\Rightarrow$ quasiconvexity:**

Jensen's Inequality: V Banach space

$f : V \to (-\infty, +\infty]$ convex, lsc

μ probability measure, $(V, \mathcal{M})$

$g \in L^1(V, \mathcal{M}, \mu)$. Then

$$\boxed{f\left(\int_V g\, d\mu\right) \leq \int_V f(g)\, d\mu}$$

<u>Proof of (*):</u>

f cx <u>claim</u> $f(\xi_0) \leq \int_Q f(\xi_0 + \nabla\varphi(x))\,dx$

$Q := (0,1)^N$ $\forall \varphi \in W_0^{1,\infty}(Q;\mathbb{R}^d)$

$$\mu := \mathcal{L}^N \llcorner Q \quad \Big(\text{notation: } \mu \llcorner A\,(E) := \mu(A \cap E), \quad A, E \in \mathcal{M} \Big)$$

$$g(x) := \xi_0 + \nabla\varphi(x)$$

Jensen's $\Rightarrow$

$$f\left(\int_Q (\xi_0 + \cancel{\nabla\varphi(x)})\, dx \right) \le \int_Q f(\xi_0 + \nabla\varphi(x))\, dx$$

$$\int_Q \nabla\varphi = \int_{\partial Q} \varphi \otimes \nu \, dH^{N-1} = 0$$

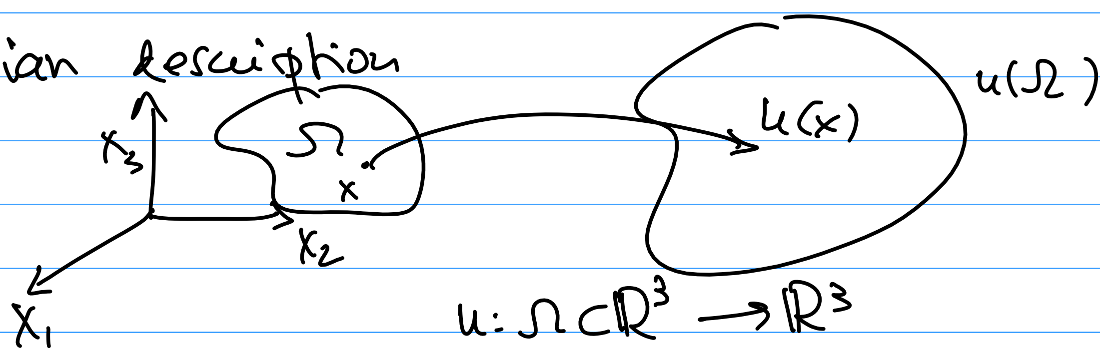
H^{N-1} ... $N-1$ Hausdorff measure ∎

We'll see: quasiconvexity $\not\Rightarrow$ convexity (unless $N=1$ or $d=1$)

$$\boxed{\begin{array}{c} \text{convex} \Rightarrow \text{polyconvex} \Rightarrow \text{quasiconvex} \Rightarrow \text{rank-1 convex} \\ \not\Leftarrow \qquad\qquad \not\Leftarrow \qquad\qquad \not\Leftarrow \text{ open if } d=2,\ N \ge 2 \end{array}}$$

In nonlinear elasticity, convexity is the <u>wrong</u> hyp.

Lagrangian description



$$u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$$

$f : \mathbb{R}^{3\times 3} \longrightarrow [0, +\infty]$ bulk/stored energy density

Total energy $\quad I(u) := \int_\Omega f(\nabla u(x))\,dx$

- To avoid interpenetration of matter, u needs to be invertible + orientation preserving

$$\Downarrow$$

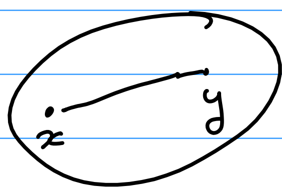
$$\det \nabla u > 0 \qquad \text{a.e.}$$

$$f(\xi) \to +\infty \quad \text{as } \det \xi \to 0^+$$

Incompatible with convexity!

$\{\xi \in \mathbb{R}^{3\times 3} : \det \xi > 0\}$ is not a convex set

Def V vector space, $C \subset V$ is convex if $\forall x, y \in C$, $\theta \in (0,1) \Rightarrow$



$$\theta x + (1-\theta) y \in C$$

$$\theta = 1/2, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det \mathbb{1}, \det A > 0, \qquad \det(\theta \mathbb{1} + (1-\theta) A) < 0$

f cx $+$ [$f(\xi) = +\infty$ iff $\det \xi \le 0$]

$\Downarrow$ impossible

- still in non-linear elasticity, expect f to be frame-indifferent

$$f(\xi) = f(R\xi) \quad \forall \xi \in \mathbb{R}^{3\times3}, \; R \in SO(3)$$

$$SO(N) := \{ R \in \mathbb{R}^{N\times N} : R^T R = \mathbb{1}, \; \det R = 1 \}$$

- In addition, crystallographic symmetry

$$f(\xi) = f(\xi R), \quad \forall \xi \in \mathbb{R}^{3\times3}, \; R \in \mathcal{S}$$

$\mathcal{S}$ subgroup of $SO(N)$

no preferred orientations $\to$ isotropic

Polar Decomposition : $\xi \in \mathbb{R}^{N\times N}$

$$\det \xi > 0 \Rightarrow \xi = RU, \quad U = \sqrt{\xi^T \xi}$$

$$R \in SO(N), \quad U = U^T > 0 \quad (Ux \cdot x > 0, \; x \neq 0)$$

$C := \xi^T \xi$ ... right Cauchy-Green strain tensor

$U := \sqrt{C}$ ... right stretching tensor

frame indifference: $f(\xi) = f(\tilde{R}\xi) \quad \forall \tilde{R}$

$(\tilde{R} = R^T) \quad \Rightarrow f(\xi) = f(U)$

frame ind. + isotropy $\Longrightarrow$

$$f(\xi) = \phi(v_1, v_2, v_3) \qquad \Phi \text{ symmetric}$$

$v_i \dots$ principal stretches
eigenvalues of U

<u>Example</u> **Ogden rubber-like materials**

$$f(\xi) = \phi(v_1, v_2, v_3) := \sum_{i=1}^{M} a_i \, \varphi(\alpha_i) + \sum_{i=1}^{P} b_i \, \psi(\beta_i) + \underbrace{h(v_1 v_2 v_3)}_{\det \xi}$$

$$\varphi(\alpha) := v_1^{\alpha} + v_2^{\alpha} + v_3^{\alpha}$$

$$\psi(\beta) := (v_2 v_3)^{\beta} + (v_1 v_2)^{\beta} + (v_3 v_1)^{\beta} - 3$$

$a_i, b_i > 0 \qquad \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_M \geq 1$
$\qquad\qquad\qquad\quad \beta_1 \geq \beta_2 \geq \dots \geq \beta_P \geq 1$

$h : [0, +\infty) \to [0, +\infty)$ convex, $h(\delta) \to +\infty$ as $\delta \to 0^+$

If $\alpha_1 \geq 2$, $\beta_1 \geq 3/2$ then

- f is <u>quasiconvex</u> (polyconvex)
- f is <u>coercive</u>: $f(\xi) \geq c\,(|\xi|^p + |\mathrm{cof}\,\xi|^q) - 1/c$

$c > 0$

$p \geq 2, \quad q \geq 2$

PLAN OF COURSE

I. Integrands $f = f(\nabla u)$
- space $W^{1,p}(\Omega;\mathbb{R}^d)$
- well-posedness of $u \mapsto \int_\Omega f(\nabla u)\,dx$
- strong conv. in $W^{1,p}$
- weak conv. in $W^{1,p}$ – nec. cond.

II. Quasiconvex Functions and Envelopes

III. More of weak conv. in $W^{1,p}$ – suff. cond.
- vectorial case
- scalar case

IV. Relaxation

V. Integrands $f = f(x, \nabla u)$

VI. Integrands $f = f(u, \nabla u)$

VII. Integrands $f = f(x, u, \nabla u)$

VIII. Applications