Calc. Var. to understand existence, uniqueness, regularity and other properties of energy functionals

$$\boxed{u \longmapsto \int_\Omega f(x, u(x), \nabla u(x))\, dx}$$

$\Omega \subset \mathbb{R}^N$ open, connected, $N \geq 1$

$u: \Omega \to \mathbb{R}^d$, $d \geq 1$

$u = (u_1, \dots, u_d)$, $u_i : \Omega \to \mathbb{R}$

∇u ... $d \times N$ matrix

$$\nabla u = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_N} \\ \vdots & & \\ \frac{\partial u_d}{\partial x_1} & \cdots & \frac{\partial u_d}{\partial x_N} \end{pmatrix}$$

Motivation

- materials science: thin structures
  - multiscale problems
    - homogenization
    - microstructures
  - thin structures
  - phase transitions
  - fracture + damage $\rightsquigarrow$ BV
  - ...
- image processing: recolorization, denoising, ... $\rightsquigarrow$ BV

Central Problem in Calc. Var

existence, uniqueness, regularity properties of solutions of the minimization problem

$$\min \{ I(u) : u \in \mathcal{A} \}$$

$$\boxed{I(u) := \int_\Omega f(x, u(x), \nabla u(x))\, dx}$$

$\mathcal{A}$ ... class of admissible fields u

e.g. constrained subset of a Sobolev space

EXISTENCE

"guaranteed" under suitable

- "convexity" properties of f
- lsc properties of $I(\cdot)$, i.e.

$u_n \to u \Rightarrow I(u) \le \underline{\lim}\, I(u_n)$

lower semicontinuity

Tonelli's Direct Method in Calc. Var.

V normed space

$I : V \to [-\infty, +\infty]$, $I \not\equiv +\infty$,

$\mathcal{A} \subset V$

Step 1 Consider a minimizing (infimizing) sequence $\{u_n\} \subset \mathcal{A}$, i.e.,

$$\inf \{ I(u) : u \in \mathcal{A} \} = \lim_{n \to +\infty} I(u_n)$$

step 2 wlog $\sup_n I(u_n) < +\infty$

$+$ coercivity

$\Downarrow$

$\exists$ subsequence $\{u_{n_k}\} \subset \{u_n\}$, $u_{n_k} \to u_0$

some $u_0 \in V$

A is weakly closed, so that $u_0 \in A$

candidate for solution of minimization problem

Step 3 Establish swlsc (sequential weakly lower semicontinuity) of $I(\cdot)$, i.e.,

challenge!

$$v_n \rightharpoonup v \quad \Rightarrow \quad I(v) \leq \underline{\lim}\, I(v_n)$$

Step 4 Conclude that

$$I(u_0) = \min_{u \in A} I(u)$$

Why? $$\inf_{u \in A} I(u) \leq \boxed{I(u_0)} \underset{(2)+(3)}{\leq} \underline{\lim}\, I(u_{n_k})$$

$$\underset{(1)}{=} \inf_{u \in A} I(u)$$

Example: $V = W^{1,p}(\Omega; \mathbb{R}^d)$, $1 < p < +\infty$

$\Omega \subset \mathbb{R}^N$ open, bounded, Lipschitz

$u \in W^{1,p}$: $u \in L^p$, i.e., $\underbrace{\left(\int_\Omega |u|^p dx\right)^{1/p}}_{=:\|u\|_p} < +\infty$

$Du \in L^p$,

$$\|u\|_{W^{1,p}} = \|u\|_{1,p} := \|u\|_p + \|Du\|_p$$

Fix $\bar{u} \in W^{1,p}(\Omega; \mathbb{R}^d)$

$$A := \bar{u} + W_0^{1,p}(\Omega; \mathbb{R}^d)$$

$$f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \longrightarrow \mathbb{R}$$

$$(x, u, \xi) \longmapsto f(x, u, \xi)$$

f is Carathéodory:

- $f(x,\cdot,\cdot)$ continuous for a.e. $x \in \Omega$
- $f(\cdot, u, \xi)$ measurable for all $(u,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N}$

f is coercive:

$$f(x, u, \xi) \geq c\, |\xi|^p - \frac{1}{c}, \quad c > 0$$

a.e. $x \in \Omega$, all $(u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N}$

$I(\cdot) \not\equiv +\infty$

step 1 $\inf \{ I(u) : u \in A \} = \lim \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx$
$\quad u_n \in A$

step 2 coercivity $\Rightarrow c \int_\Omega |\nabla u|^p - \frac{1}{c} |\Omega| \leq \int_\Omega f(x, u(x), \nabla u(x))\, dx$

$$\begin{cases} \left\{ \int_\Omega |\nabla u_n|^p \right\} \text{ is bounded} \\ u_n \in \bar{u} + W_0^{1,p}(\Omega; \mathbb{R}^d) \end{cases}$$

Poincaré's Inequality $\Rightarrow \{u_n\}$ bdd in $W^{1,p}$

$1 < p < +\infty$

$\Rightarrow \exists$ subsequence $\{u_{n_k}\}$ s.t.

$$\left| \begin{array}{l} u_{n_k} \rightharpoonup u_0, \text{ some } u_0 \in W^{1,p} \\ u_0 \in A \end{array} \right.$$

step 3 Need to show that

$v_n \rightharpoonup v$ $W^{1,p}$ $\Rightarrow \int_\Omega f(x, v, \nabla v) \leq \underline{\lim} \int_\Omega f(x, v_n, \nabla v_n)$

Step 4 Direct Method $\Rightarrow I(u_0) = \min_{u \in A} I(u)$

QUESTION 1

Is it possible to find necessary and/or sufficient conditions on f guaranteeing swlsc of $u \mapsto I(u)$

related to some "convexity" properties of $f(x, u, \bullet)$

recall from Funct. Anal.

Thm X Banach space

$I : X \to \mathbb{R} \cup \{+\infty\}$ convex and (strongly) lsc. Then I is swlsc.

Def I is convex if

$I(\theta u + (1-\theta) v) \leq \theta I(u) + (1-\theta) I(v)$

for all $\theta \in (0,1)$, u, v

Corollary $f : \mathbb{R}^{d \times N} \to [0, +\infty)$ convex

$1 < p < +\infty$, $I : u \in W^{1,p}(\Omega; \mathbb{R}^d) \to \int_\Omega f(\nabla u(x)) dx$

Then I is swlsc.

Proof f convex $\Rightarrow$ I is convex

Can use Thm if I is strongly lsc in $W^{1,p}$.

$u_n \to u$ $W^{1,p}$ $\Rightarrow$ $\nabla u_n \to \nabla u$ L^p

wlog $\nabla u_n(x) \to \nabla u(x)$ a.e. $x \in \Omega$

$f \geq 0$ + Fatou's lemma:

$$\underline{\lim} \int_\Omega f(\nabla u_n) \geq \int_\Omega \underline{\lim}\, f(\nabla u_n) = \int_\Omega f(\nabla u)$$

cont. ∎

swlsc.

Recall nec + suff conditions for swlsc. in the unconstrained case, i.e.

$$v \in L^p \longmapsto \int f(x, v(x))\, dx$$

~~(our case: $v(x) = (\underbrace{u(x)}_{d}, \underbrace{Du(x)}_{d \times N})$)~~

F + Leoni, Springer 2007

Thm $E \subset \mathbb{R}^N$ Lebesgue measurable set, $1 \le p < +\infty$

$f : E \times \mathbb{R}^m \to (-\infty, +\infty]$ $\mathcal{L}^N \times \mathcal{B}$ measurable

(H1) $f(x, \cdot)$ lsc in $\mathbb{R}^m$ a.e. $x \in E$

(H2) $f(x, v) \geq -c|v|^p - \gamma(x)$ a.e. $x \in E$, $\forall v \in \mathbb{R}^m$

some $c > 0$, $\gamma \in L^1(E)$

Then $v \in L^p(E; \mathbb{R}^m) \longmapsto \int_E f(x, v(x))\, dx$

is swlsc in L^p IFF

(i) $f(x, \cdot)$ convex in $\mathbb{R}^m$, a.e. $x \in E$

(ii) $f(x, v) \geq a(x) + b(x) \cdot v$ a.e. $x \in E$, $\forall v \in \mathbb{R}^m$

some $a \in L^1(E)$, $b \in L^{p'}(E; \mathbb{R}^m)$

$\frac{1}{p} + \frac{1}{p'} = 1$ (conjugate exponent)

Rmk H1 + H2 are not restrictive

(H1) $f(x, \cdot)$ lsc in $\mathbb{R}^m$ a.e. $x \in E$,

$y_n \to y$ in $\mathbb{R}^m \Rightarrow f(x, y) \leq \underline{\lim}\, f(x, y_n)$

This is a necessary condition for strong lsc of $v \in L^p \to \int_E f(x, v(x))\, dx$

(H2) $f(x, v) \geq -c|v|^p - \gamma(x)$

nec cond for

$$\int_E f(x, v(x))\, dx \text{ to be well defined}$$

i.e., $\left\{ \int_E f^+(x, v(x))\, dx, -\int_E f^-(x, v(x))\, dx \right\} \neq \{+\infty, -\infty\}$

$f^- := -\min\{f, 0\}$

$f^+ := \max\{f, 0\}$

$\Rightarrow f = f^+ - f^-$

Constrained case : $E = \Omega$ open connected

$\operatorname{curl} v = 0$, $v = \nabla u$

(H2) → **Thm** $1 \leq p < +\infty$, $|\Omega| < +\infty$,

$f : \mathbb{R}^{d \times N} \to (-\infty, +\infty]$ Borel function

If for all $u \in W^{1,p}(\Omega; \mathbb{R}^d)$, $\int_\Omega (f(\nabla u(x)))^- dx < +\infty$

IFF (c) $f(\xi) \geq -c(|\xi|^p + 1)$, $c > 0$ if $1 \leq p < +\infty$

(c) f locally bounded from below if $p = +\infty$

if $K \subset\subset \mathbb{R}^{d \times N}$ (compactly contained)

$\inf_{\xi \in K} f(\xi) > -\infty$

Thm $|\Omega| < +\infty$, open connected subset of $\mathbb{R}^N$

$1 \leq p \leq +\infty$, $f : \mathbb{R}^N \to (-\infty, +\infty]$ lsc

- $f(\xi) \geq -C(|\xi|^p + 1)$ $\forall \xi \in \mathbb{R}^N$, $1 \leq p < \infty$, $C > 0$
- f locally bounded from below if $p = +\infty$

Then $u \in W^{1,p}(\Omega) \to \int_\Omega f(\nabla u(x))\,dx$ is

swlsc in $W^{1,p}$ (w-* if $p = +\infty$) IFF f convex

N.B. This is the scalar-valued case

$u : \Omega \to \mathbb{R}$, i.e., $d = 1$, $\nabla u \in \mathbb{R}^N$

Rmk This thm ~~is~~ may be false if $|\Omega| = +\infty$