Fokker-Planck Equation

$$\Phi_\lambda(t) = \lambda \int_\Omega \rho \log \frac{\rho}{\rho_\lambda} dx = F_\lambda(\rho) + \text{const}$$ Kullback-Leibler relative entropy

$$= \lambda \int_\Omega (\psi_\lambda \rho + \rho \log \rho) dx$$ $$\underbrace{\int_\Omega e^{-\psi_\lambda} dx = 1}_{\text{Kraft Inequality}}$$

$\inf \Phi_\lambda(t_\infty) \underset{\text{convex}}{\sim} \uparrow$ dual problem

finding σ analogous to prefix code with maximal information

associated Fisher information

Properties of F-P

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\sigma \frac{\partial \rho}{\partial x} + \psi' \rho\right) \quad \text{in } \Omega,\ t>0$$

$$\sigma \frac{\partial \rho}{\partial x} + \psi' \rho = 0 \quad \text{in } \partial\Omega,\ t>0$$

or ρ periodic

$$\int_\Omega \rho\, dx = 1, \quad \rho \geq 0$$

$$\rho^\#(x) = \frac{1}{Z} e^{-\frac{\psi(x)}{\sigma}}$$ stationary distribution, $\int_\Omega \rho^\# dx = 1$

Want to show: $\Phi_\sigma \longrightarrow 0$ exponentially fast as $t \to \infty$

- using log-Sobolev inequality
- conventional problem in PDE

$$\int_\Omega \rho_t \zeta\, dx = \int_\Omega (\sigma \rho_x + \psi' \rho)_x \zeta\, dx$$

$$= -\int_\Omega (\sigma \rho_x + \psi' \rho) \zeta_x\, dx$$

$$= -\sigma \int_\Omega \left(\rho_x + \frac{\psi'}{\sigma}\rho\right) \zeta_x \, dx$$

$$= -\sigma \int_\Omega e^{-\frac{\psi}{\sigma}} \left(e^{\frac{\psi}{\sigma}}\rho\right)_x \zeta_x \, dx$$

$\Longrightarrow$

$$\int_\Omega \left(\frac{\rho}{\rho^\#}\right)_t \zeta \, dx = -\sigma \int_\Omega \left(\frac{\rho}{\rho^\#}\right)_x \zeta_x \, \rho^\# \, dx$$

Now let

$$u(x,t) = \left(\frac{\rho}{\rho^\#}\right)(x,t), \qquad a = \rho^\# > 0 \quad \text{in } \Omega,\ t > 0$$

$$a u_t = \sigma (a u_x)_x \qquad \text{in } \Omega$$

φ convex, nonnegative

$$\Phi = \int_\Omega \varphi\left(\frac{\rho}{\rho^\#}\right) \rho^\# \, dx = \int_\Omega \varphi(u) \, a \, dx$$

$$\Phi'(t) = \frac{d}{dt} \int_\Omega \varphi(u) \, a \, dx$$

$$= \int_\Omega \varphi'(u) \frac{\partial u}{\partial t} a \, dx$$

$$= \sigma \int_\Omega \varphi'(u) (a u_x)_x \, dx$$

$$= -\sigma \int_\Omega \varphi''(u) \, a \, u_x^2 \, dx \quad < 0$$

$\Longrightarrow$ Φ decreasing

$$\Phi(0) - \Phi(\infty) = \sigma \int_0^\infty \int_\Omega \varphi''(u) \, u_x^2 \, a \, dx \, dt < +\infty$$

$$\Longrightarrow \quad \int_\Omega \varphi''(u) \, u_x^2 \, a \, dx \longrightarrow 0 \quad \text{as } t \longrightarrow \infty$$

Choose special φ's :

$$\varphi(\xi) = \frac{1}{2}(\xi - 1)^2 \qquad \varphi'' = 1$$

$$\Longrightarrow \quad \int_\Omega u_x^2 \, a \, dx \longrightarrow 0$$

$\Rightarrow \quad u \longrightarrow \text{constant} = 1, \quad \text{as } t \to \infty$

$\Rightarrow \quad \rho \longrightarrow \rho^{\#} \quad \text{as } t \to \infty$

In particular, whenever $\varphi(1) = 0$, have that $\Phi(\infty) = 0$

$$\varphi(\xi) = \xi \log \xi$$

$$\Phi(t) = \int_\Omega \rho \log \frac{\rho}{\rho^{\#}}\, dx = \sigma \int_\Omega u \log u \, a \, dx$$

rate at which tends to 0

Note Poincaré: $\displaystyle\int_\Omega \zeta a \, dx = 0 \Rightarrow$

$$\int_\Omega \zeta^2 a \, dx \leq C \int_\Omega \zeta_x^2 a \, dx$$

prove by assuming it is false

$$V(t) = \frac{1}{2} \int_\Omega (u-1)^2 a \, dx \quad \text{for which} \quad \int_\Omega (u-1)\, a \, dx = 0$$

$$\frac{dV}{dt} = -\sigma \int_\Omega u_x^2 a \, dx$$

$$\leq -\frac{\sigma}{C_0} \int_\Omega (u-1)^2 a \, dx$$

$$= -\frac{2\sigma}{C_0} V$$

$$\Rightarrow \quad V(t) \leq V(0) e^{-\varepsilon t} \qquad 0 < t < \infty \qquad (\varepsilon > 0)$$

$$\varphi(\xi) = \xi \log \xi, \qquad \varphi''(\xi) = \frac{1}{\xi} > 0$$

$$\frac{d\Phi}{dt} = -\sigma \int_\Omega \frac{1}{u} u_x^2 \, a \, dx$$

u bounded below $\Rightarrow$ can find a $\delta > 0$ small enough s.t.

$$\frac{d}{dt}\left(V - \delta \Phi\right) = -\sigma \int_\Omega u_x^2 \left(1 - \frac{\delta}{u}\right) a \, dx < 0$$

$$V(\infty) = \Phi(\infty) = 0$$

$$\Phi(t) \leq \frac{1}{\delta} \mho(t) \leq \frac{1}{\delta} \mho(0) e^{-\varepsilon t}, \quad 0 < t < +\infty$$

establishes exponential decay of relative entropy.