Wasserstein metric $\quad D \subset \mathbb{R}$

f, f^* probability densities

(1) $d(f,f^*)^2 = \inf_{p} \int_{D \times D} |x-y|^2 \, dp(x,y)$ $\quad p$ = joint distributions for f, f^*

$f, f^* > 0 \implies \exists$ transfer map

$\phi : D \to D$

$\int_D \zeta f \, dy = \int_D \zeta(\phi(x)) f^*(x) \, dx \qquad \zeta \in C(\bar{D})$

$\phi(x) = F^{*-1}(F(x)), \; x \in D,$

$F(x) = \int_{-\infty}^{x} f(x')\,dx', \quad F^*(x) = \int_{-\infty}^{x} f^*(x')\,dx'$

(2) $d(f,f^*)^2 = \int_D |x-\phi(x)|^2 f^* \, dx$

$\frac{1}{\tau} d(f,f^*)^2 = \inf \int_0^\tau \int_D v^2 f \, d\xi \, dt$

over deformation paths $f(\xi,t)$ subject to

$f_t + (vf)_\xi = 0 \qquad$ (continuity)

$f(\xi,0) = f^*(\xi), \quad f(\xi,\tau) = f(\xi) \qquad$ (initial + terminal)

$\left(\begin{array}{l} v_t + v v_x = 0 \qquad \text{(Burgers)} \quad \text{for extremal pair } (f,v) \\ \frac{d}{dt} d(f,f^*)^2 = 0 \quad \text{on extremal} \end{array} \right)$

can show by Schwarz inequality

return to dissipation inequality

$\mu_0 \int_0^\tau \int_\Omega \left(\frac{\partial \rho}{\partial t}\right)^2 dx\,dt \rightsquigarrow \int_0^\tau \int_\Omega v^2 f \, dx\,dt$

for some f

$f_t + (vf)_x = 0$

$f|_{t=t_0} = \rho|_{t=t_0}, \; f|_{t_0+\tau} = \rho|_{t_0+\tau}$

make an estimate

$\Omega = (0,1) \qquad \rho(x,t) \geq \delta > 0 \quad \text{in } \Omega,\ t>0$

claim

$$\frac{1}{\tau} d(\rho,\rho^*)^2 \leq \int_0^\tau \int_\Omega v^2 \rho \, dx\, dt \leq \frac{c_\Omega}{\min \rho} \int_0^\tau \int_\Omega \left(\frac{\partial \rho}{\partial t}\right)^2 dx\, dt$$

$\rho^* = \rho|_{t_0},\ \rho = \rho|_{t_0+\tau}$

(v,ρ) integrate continuity equation: $\quad 0<t<\tau$

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} = \underbrace{\frac{\partial F}{\partial t} + v\rho}_{} = c \quad \text{in } \Omega$$

$$F_t\big|_{t=0} = 0, \quad v\big|_{\partial\Omega} = 0 \implies c = 0$$

$$v^2\rho(x,t) = \frac{F_t^2}{\rho} \leq \frac{x}{\min \rho}\int_\Omega \rho_t(y,t)^2 dy \qquad x\in\Omega,\ t>0$$

Schwarz

$$\int_0^\tau \int_\Omega v^2\rho\, dx\, dt \leq \int_0^\tau \int_\Omega \frac{x}{\rho(x,t)} dx \int_\Omega \rho_t(y,t)^2 dy\, dt$$

$$\leq \frac{c_\Omega}{\min \rho} \int_0^\tau \int_\Omega \rho_t^2\, dy\, dt$$

replace by

in inequality

$\implies$

$$\mu \int_0^\tau \int_\Omega v^2 \rho\, dx\, dt + F_\lambda(\rho)\Big|_{t=\tau} \leq F_\lambda(\rho)\Big|_{t=0}$$

assume now that our path is the most likely to occur

$$\frac{\mu}{2\tau} d(\rho,\rho^*)^2 + F_\lambda(\rho) = \inf_\eta \left\{ \frac{\mu}{2\tau} d(\eta,\rho^*)^2 + F_\lambda(\eta) \right\}$$

$\implies$ for each relaxation time, have a ρ

$\Rightarrow$ as $\varepsilon \to 0$

$$\mu \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \alpha}\left(\lambda \frac{\partial \rho}{\partial \alpha} + \psi' \rho\right) \quad \text{in } \Omega$$

Fokker-Planck

$\lambda = ?$ entropy method!