Modeling "collisions".

Rarefied gas, large number of particles.

\( f(t, x, \varepsilon) \) - density of particles at time \( t \) and point \( x \) with speed \( \varepsilon \).

No collisions: \( \frac{\partial f}{\partial t} + \varepsilon \cdot \nabla_x f = 0 \)

With collisions: \( \frac{\partial f}{\partial t} + \varepsilon \cdot \nabla_x f = Q(f) \)  

Collision term

Elastic case: hard spheres same radius

(a) \( \rightarrow \)  \( \leftarrow \)  \( \leftrightarrow \)  velocities are exchanged
(b) in $\mathbb{R}^3$ - angle of collision is a unit vector $n$ parallel to axis joining centers at collision.

Components of velocities $\perp n$ will be preserved, components $\parallel n$ will be exchanged.

\[
\begin{align*}
\hat{\xi}' &= \hat{\xi} - (n \cdot (\hat{\eta} - \hat{\xi})) n \\
\hat{\zeta}' &= \hat{\zeta} + (n \cdot (\hat{\xi} - \hat{\zeta})) n
\end{align*}
\]

Exchange in $\perp n$ direction.

New velocities:\[
\begin{align*}
\hat{\zeta}' \cdot n &= \hat{\xi} \cdot n \\
\hat{\xi}' \cdot n &= \hat{\zeta} \cdot n
\end{align*}
\]
Deriving Boltzmann eqn:

Rate of collisions at an angle \( \mathbf{n} \) is

\[ \frac{1}{n \cdot \mathbf{V} \, d\mathbf{n}} \]

\[ \Rightarrow \ldots \Rightarrow \quad \frac{e^{-z_t}}{\pi} \]

Loss term:

\[ Q_-(\mathbf{f})(\mathbf{z}) = \alpha \int \int \int_{S_2} \int_{S_2^*} \mathbf{n} \cdot (\mathbf{z} - \mathbf{z}_t) |f(\mathbf{z})f(\mathbf{z}_t^*)| \, d\mathbf{n} \, d\mathbf{z}_t^* \, d\mathbf{z}_t^* \]

Elastic collision is reversible so

\[ |\mathbf{n}(\mathbf{z} - \mathbf{z}_t)| \, f(\mathbf{z})f(\mathbf{z}_t^*) = |\mathbf{n}(\mathbf{z} - \mathbf{z}_t)| f(\mathbf{z})f(\mathbf{z}_t^*) \]

\[ \Rightarrow \text{Gain term:} \]

\[ Q_+ f(\mathbf{z}) = \alpha \int \int \int_{S_2} \int_{S_2^*} \mathbf{n} \cdot (\mathbf{z} - \mathbf{z}_t) |f(\mathbf{z})f(\mathbf{z}_t^*)| \, d\mathbf{n} \, d\mathbf{z}_t^* \, d\mathbf{z}_t^* \]

Boltzmann eqn:

\[ \partial_t f + \mathbf{z} \cdot \nabla f = \int \int \int_{S_2} (f(\mathbf{z}') - f(\mathbf{z}_t^*)) |\mathbf{n}(\mathbf{z} - \mathbf{z}_t)| \, d\mathbf{n} \, d\mathbf{z}_t^* \, d\mathbf{z}_t^* \]
**H-theorem:**

Rewrite Boltzmann as

\[
\frac{\partial}{\partial t} \mathcal{H}(t, x) + \nabla_x \cdot \mathbf{J}(t, x) = W(t, x)
\]

where \( \mathcal{H}(t, x) = \int f \ln f \, d\mathbf{\xi} = -S \)

\[
W = \int Q(f) \ln f \, d\mathbf{\xi}
\]

Since \( W \leq 0 \) and \( f \) decays at \( |x| \to \infty \)

\[
\frac{d}{dt} \mathcal{H}(t) \leq 0
\]

Max entropy solution is a Gaussian distribution
Beyond elastic case:

Granular flow

\[ v_1^* - v_2^* = -r (u_1 - u_2) \]

\( u_i \) \( u_j \) \( u_k \) \( u_l \)

new vel. \( \uparrow \)
old vel. \( \uparrow \)

\( r = \begin{cases} 0 & \text{inelastic case} \\ 1 & \text{elastic case} \end{cases} \)

\((u_1, u_2) \rightarrow (pu_1 + qu_2, qu_1 + pu_2)\)

\[ p + q = 1, \quad r = 1 - 2p \]

\[ K(v_1, v_2) = |v_1 - v_2|^2 \]

collision rate

\[ \lambda = 0 \Rightarrow f(v) \rightarrow S(v) \]

\[ T = T_0 e^{-\frac{1}{2}t} \]

exp. decay of temperature

\[ \gamma_n = 1 - p^n - q^n \]
Elastic | Granular | Smoluchowski

Mass = const | Mass = const | Mean mass grows

$E_{\text{Kinetic}} = \text{const}$ | $E_{\text{Kin}} \rightarrow$ | $E_{\text{Kin}} \downarrow$ (cooling effect)

Reversible | Irreversible collision