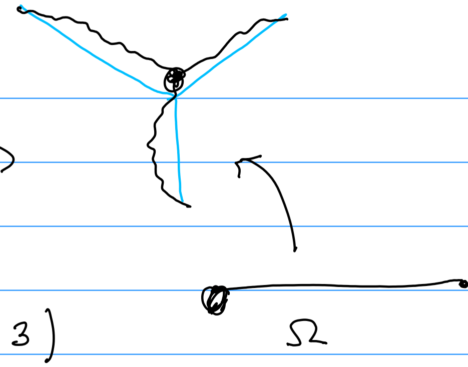
Asymptotic Stability

solution starts near equilibrium $\Rightarrow$
exists for all time



local solutions: Broussard Reitich (1993)

Set up

$$\xi^{(j)} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^2, \quad \Omega = (0,1)$$

$$v_n^{(j)} = \kappa^{(j)}, \quad j = 1, 2, 3$$

$$\sum b^{(j)} = 0 \quad \text{at TJ}$$

$$\xi^{(1)} = (u_1, u_2), \quad \xi^{(2)} = (u_3, u_4), \quad \xi^{(3)} = (u_5, u_6)$$

$$\frac{\partial u_1}{\partial t} = \frac{1}{u_{1x}^2 + u_{2x}^2} \frac{\partial^2 u_1}{\partial x^2}, \quad \frac{\partial u_2}{\partial t} = \frac{1}{u_{1x}^2 + u_{2x}^2} \frac{\partial^2 u_2}{\partial x^2} \quad \text{in } \Omega, \ t > 0$$

$$\frac{\partial u_3}{\partial t} = \ldots \qquad \frac{\partial u_4}{\partial t} =$$

$$\frac{\partial u_5}{\partial t} = \sim \qquad \frac{\partial u_6}{\partial t} =$$

System

$$\frac{u_{1x}}{\sqrt{u_{1x}^2 + u_{2x}^2}} + \frac{u_{3x}}{\sqrt{u_{3x}^2 + u_{4x}^2}} + \frac{u_{5x}}{\sqrt{u_{5x}^2 + u_{6x}^2}} = 0$$

$$\frac{u_{2x}}{\sqrt{\quad}} + \frac{u_{4x}}{\sqrt{\quad}} + \frac{u_{6x}}{\sqrt{\quad}} = 0$$

Herring

TJ

$$u_1 = u_3 = u_5 \text{ and } u_2 = u_4 = u_6 \quad \text{at } x = 0$$

$$u_i = c_i \quad \text{at } x = 1, \ t > 0$$

$$v = \text{stationary solutions}; \quad v_{1x}^2 + v_{2x}^2 = 1$$

$$z = u - v$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u_1 = \left(\frac{1}{u_{1x}^2 + u_{2x}^2} - 1\right) \frac{\partial^2 u_1}{\partial x^2}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right) z_1 = \left(\frac{1}{u_{1x}^2+u_{2x}^2}-1\right)\frac{\partial^2 z_1}{\partial x^2} = f_1(z, z_x, z_{xx})$$

$$|f_j| \leq C \left|\frac{\partial z}{\partial x}\right| \left|\frac{\partial^2 z}{\partial x^2}\right|$$ quadratic

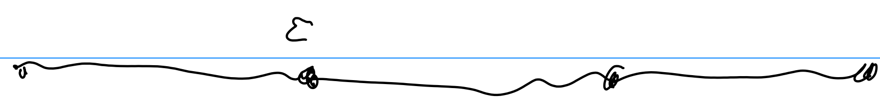
$$Bz = g\left(z, \frac{\partial z}{\partial x}\right) :$$

$$\left|\begin{array}{ll} \frac{\partial z_1}{\partial x} + \frac{\partial z_3}{\partial x} + \frac{\partial z_5}{\partial x} = g_1 & \\ \frac{\partial z_2}{\partial x} + \frac{\partial z_4}{\partial x} + \frac{\partial z_6}{\partial x} = g_2 & \text{at } x=0,\ t>0 \\ z_1 = z_3 = z_5 & \\ z_2 = z_4 = z_6 & \end{array}\right.$$

Main Result $\exists\ \varepsilon>0$ s.t.

$$\int_\Omega z_{0xx}^2\, dx \leq \varepsilon$$

$\Rightarrow$ there is a solution of the system $0<t<\infty$



$$z = z_1 + z_2 \qquad z_1, z_2 \in \mathbb{R}^6$$

$$\left|\begin{array}{lll} \left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right) z_1 = f\left(\frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) & \text{in } \Omega,\ t>0 & \\ Bz_1 = g\left(w, \frac{\partial w}{\partial x}\right) & x=0,\ t>0 & \\ z_1 = 0 & x\in\Omega,\ t=0 & \end{array}\right.$$

inhomogeneous, 0 initial data

$$\left|\begin{array}{lll} \left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right) z_2 = 0 & \text{in } \Omega,\ t>0 & \\ Bz_2 = 0 & x=0,\ t>0 & \\ z_2 = z_0 & x\in\Omega,\ t=0 & \end{array}\right.$$

homogeneous, nonzero initial data

$$z = Tw$$

Solutions exist by known parabolic theory Solonnikov (1965) think about this yourself

$$V = \{ w = (w_1, \dots, w_6) \in H^1(\Omega) : w_1 = w_3 = w_5,\ w_2 = w_4 = w_6 \text{ at } x = 0 \}$$

$$a(z, \zeta) = \int_\Omega z_x \cdot \zeta_x \, dx$$

$$z = Tw$$

$$\sup_t \int_\Omega w_{xx}^2 \, dx < \delta, \quad \int_0^\infty \int_\Omega w_x^2 \, dx\,dt < \delta \quad \Big| \implies$$

$$\int_\Omega z_{0xx}^2 \, dx < \varepsilon$$

$$\sup_t \int_\Omega z_{xx}^2 \, dx < \delta, \quad \int_0^\infty \int_\Omega z_{xx}^2 \, dx\,dt \leq \delta$$

z_2: $$\int_\Omega z_{2xx}^2 \, dx \leq \frac{M}{2} \int_\Omega z_{0xx}^2 \, dx \leq \frac{M}{2} \varepsilon \qquad M \text{ some constant}$$

$$\frac{d}{dt} \int_\Omega z_{xx}^2 \, dx + \int_\Omega z_{xxx}^2 \, dx = 0$$

$$ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2$$

z_1 $$\frac{\partial z_1}{\partial t} - \frac{\partial^2 z_1}{\partial t^2} = f\left(\frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right)$$

quadratic

$$\sup_{0<t<T} \int_\Omega z_{1xx}^2 \, dx \leq \frac{C(T)}{2} \left(\int_\Omega w_{xx}^2 \, dx \right)^2 \leq \frac{C(T)}{2} \cdot \delta^2$$

$\implies$ for $z = z_1 + z_2$

$$\int_\Omega z_{xx}^2 \, dx \leq M\varepsilon + C(T)\delta^2 \leq \delta$$

$$M\varepsilon + C(T)\delta^2 \leq \delta, \quad \frac{1}{N-1} \leq \frac{1}{2M} \qquad 0 \leq t \leq N$$

$$\frac{d}{dt} \int_\Omega z_x^2 \, dx + \int_\Omega z_{xx}^2 \, dx \leq C_1 \int_\Omega w_x^2 \cdot w_{xx}^2 \, dx$$

$$\int_0^\infty \int_\Omega z_{xx}^2 \, dx \, dt \le cu < \delta$$

$$\int_1^N \int_\Omega z_{xx}^2 \, dx \, dt < \delta \implies$$

$$\int_\Omega z_{xx}^2 \, dx \le \frac{\delta}{N-1} \quad \text{at some } t^*$$

$$< \varepsilon$$

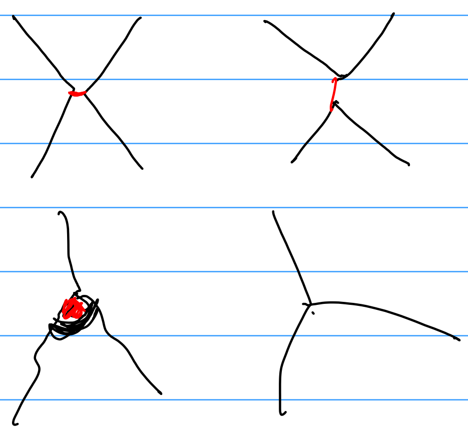
Now reproduce conditions previously imposed on w at $t=0$ for z at $t = t^* > 1$

$\implies$ can find solution in $0 < t < +\infty$ to $z = Tw$

$L^\infty(0,T, H^2(\Omega)) \times L^2(0,T, H^1(\Omega))$ any t

use ordinary compactness argument

von Below



fast unlocking

grain dilation

In the network:

are there any properties?

⟶ geometric characteristics of the network (Burke-Turnbull)

⟶ Pb electrode / 70 YA silicate

engineered by changing energy structure of boundaries texture